TWO DIMENSIONAL SUBTRACTION - TRANSFER GAMES

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Abstract. We generalize the results and conjectures in [3], showing that the nim-values of a large class of two-dimensional subtraction-transfer games are periodic. These are impartial, normal-play games with two piles of tokens, where players alternate either taking some tokens from a pile or transferring tokens from one pile to the other. In many cases, we calculate the exact period. We also develop several new notions of periodicitiy.

1. Introduction

In [3], Tamás Lengyel investigated variations of a normal-play, impartial, two-pile subtraction transfer game. All of his examples involved two piles of tokens, where a player was allowed to either subtract a particular number from the first pile, subtract another particular number from the second pile, or move a single token from the first pile to the second pile. He then proved that several games of this type are *periodic*. That is, there is a *period* $(p, q) \in \mathbb{N}^2$ so that, for any $(x, y) \in \mathbb{N}^2$, $\mathcal{SG}(x, y) = \mathcal{SG}(x+p, y) = \mathcal{SG}(x, y+q)$, where x and y denote the number of tokens in the first and second pile, respectively. In some sense, this paper merely presents a generalization of his results.

However, we also challenge a method in [2]. In this paper, Larsson, Saha, and Yokoo investigated one-, two-, and three-move vector subtraction games. In doing so, they focus only on the outcome class of game positions, ignoring nim-values. On one hand, studying only the outcome classes gives enough information for their purposes, but it is also a common point of view to consider, for a given impartial game, the calculation of outcome classes to be easier than a full calculation of nim-values. It turns out that this point of view would not have allowed us to generalize Lengyel's results, even though it 'should' be easier to consider only outcome classes. By considering outcome classes, one essentially partitions all nim-values as:

$$0 \mid 1234\ldots,$$

whereas in the study of what we are calling *Lengyel transfer games*, it is more fruitful to mentally partition nim-values according to $a \sim a \oplus 1$, that is, as:

$$01 \mid 23 \mid 45 \mid 67 \mid 89\ldots$$

Furthermore, we end up discovering some new notions of periodicity.

Definition 1.1. A *vector game* is an impartial, normal-play, two player game played on $\mathbb{N}^n$ for some n. It is determined by a set of vectors $S = \{u_i\}_{i\in I} \subseteq \mathbb{Z}^n$. A legal move consists of adding an element of S to the current game position. If each u_i is an element of $\left(\mathbb{Z}^{\leq 0}\right)^n$, we call it a *vector subtraction game*.

To ensure the game is not loopy, we further require that adding an element of S reduces some monovariant. In the case of vector subtraction games, this could be the sum of the components. In all of our examples, it will be the lexicographic order on $\mathbb{N}^2$, that is, $(x, y) \leq (x', y')$ if either $x < x'$ or both $x = x'$ and $y \leq y'$.

Definition 1.2. Define the *Lengyel transfer game* $L(b; x_1, y_1; x_2, y_2)$ as the vector game defined by $\{(0, -b), (-x_1, y_1), (-x_2, y_2)\}$, where $b, x_1, x_2 \in \mathbb{N}^+$ and $y_1, y_2 \in \mathbb{N}$.

Lengyel proved the following.

Theorem 1.3 ([3])**.**

(a) *$L(1; 1, 0; 1, 1)$ has period $(2, 2)$.*

(b) *If b is even, $L(b; b, 0; 1, 1)$ has period $(2b, 2b)$. In fact, (x, y) is a $\mathcal{P}$-position precisely when $\lfloor \frac{x+y}{b} \rfloor + x$ is even, which solves the olympiad problem which motivated [3].*

(c) *If $x_1 \geq 2$, the period of $L(1; x_1, 0; 1, 1)$ is $(2(x_1 + 1), 2)$.*

He then conjectured the following.

Conjecture 1.4 ([3]).
(a) If b is odd and at least 3 , *$L(b; b, 0; 1, 1)$ has period $(2b(b + 1), 2b)$.*
(b) For $b \geq 2$, $L(b; 1, 0; 1, 1)$ has period $(4b, 2b)$

We will generalize Theorem 1.3 and Conjecture 1.4 together as follows.

Theorem 1.5 (First Main Theorem).
(a) $L(1; 1, 0; 1, 1)$ has period $(2, 2)$.
(b) If b is even, and if x_1 is an odd multiple of b, then $L(b; x_1, 0; 1, 1)$ has period $(2b, 2b)$.
(c) In all other cases, $L(b; x_1, 0; 1, 1)$ has period $(2b(x_1 + 1), 2b)$.

As an example to aid visualization, Figures 1 and 2 show the array of NIM-values for $L(1; 1, 0; 1, 1)$ and $L(2; 3, 0; 1, 1)$. We can think of the game as being played on an infinite $\mathbb{N} \times \mathbb{N}$ board, where the origin is in the top left corner. In the latter example, a legal move is to jump up 2 units, left 3 units, or down-left 1 unit ($\sqrt{2}$ units?). To aid clarity, we will freely conflate this table of NIM-values with the game board itself. Note that the initial column and row are each numbered 0, as they represent empty piles of tokens.

0	2	0	2
1	3	1	3
0	2	0	2
1	3	1	3

FIGURE 1. Four periods of the array of NIM-values for $L(1; 1, 0; 1, 1)$

0	1	1	2	0	0	1	1	1	0	0	2	1	1	0	0
0	0	1	1	1	0	0	2	1	1	0	0	0	1	1	2
1	0	0	3	1	1	0	0	0	1	1	3	0	0	1	1
1	1	0	0	0	1	1	3	0	0	1	1	1	0	0	3

FIGURE 2. One period of the array of NIM-values for $L(2; 3, 0; 1, 1)$

In Section 2, we carefully develop the notion NIM-periodicity. Our first result in that section will imply the following.

Lemma 1.6. *The second component of the period of the Lengyel transfer game $L(b; x_1, y_1; x_2, y_2)$ is always $2b$.*

While Lemma 2.5 says that all Lengyel transfer games are *eventually* periodic in the first component.

In Section 3, we extend a key lemma of [2], and use it to develop the notion of diagonal periodicity (Definition 3.4). In Section 4, we prove Theorem 1.5. Section 5 shows that we may have arbitrarily long preperiods. In Section 6, we organize some results which simplify the remaining sections. In Section 7, we prove that in most cases, eventual diagonal periodicity is obtained, leading to the following.

Theorem 1.7 (Second Main Theorem). *The first component of the period of the Lengyel transfer game $L(b; x_1, y_1; x_2, y_2)$ is a factor of*

$$g(b, x_1, y_1, x_2, y_2) = \frac{2b(x_1 + x_2)}{\gcd\{2b, y_1 + y_2\}}.$$

Note that $g(b, x_1, 0, 1, 1) = 2b(x_1 + 1)$, matching Theorem 1.5(c).

In Section 8 we treat the periods of all games of the form $L(b; x_1, 0, x_2, y_2)$, though we must leave the final result as conjecture. In Section 9 we prove the remaining conjecture from [3], which is about a class of games with multiple transfer options. In Section 10, we explore some examples which lead to open questions.

Remark. We extensively used computer models to find our results. In doing so, if the period of $L(b; x_1, y_1; x_2, y_2)$ appears to be (p, q), then to verify it as such, we only need to compute a block of size $(2m) \times (2n)$, where $m = \max\{p, x_1, x_2\}$ and $n = \max\{q, b, y_1, y_2\}$. Hence, Figure 1 constitutes a proof of Theorem 1.5(a).

2. Nim-Periodicity

Lemma 1.6 is a direct consequence of a more subtle phenomenon.

Lemma 2.1 (Nim-Periodicity). *For any* $L(b; x_1, y_1; x_2, y_2)$, $\mathcal{SG}(x, y + b) = \mathcal{SG}(x, y) \oplus 1$.

For example, note that $\mathcal{SG}(x, y + 2) = \mathcal{SG}(x, y) \oplus 1$ for all (x, y) in Figure 2, and $\mathcal{SG}(x, y + 1) = \mathcal{SG}(x, y) \oplus 1$ for all (x, y) in Figure 1.

Note that the following proof can be generalized to any vector game of the form $\{(0, -b), (-x_i, y_i)_{i\in I}\}$, where each $x_i > 0$, each $y_i \geq 0$, and I is finite.

Proof. Consider some $L(b; x_1, y_1; x_2, y_2)$, and suppose $\mathcal{SG}(x, y) = a$. We proceed by induction on the lexicographic order of (x, y). Let S be the subset of $\{(x - x_1, y + y_1), (x - x_2, y + y_2)\}$ whose components are all nonnegative. Define $\mathcal{SG}(S)$ as the corresponding set of values. Define $S + b$ as the corresponding subset of $\{(x - x_1, y + y_1 + b), (x - x_2, y + y_2 + b)\}$, and similarly for $\mathcal{SG}(S + b)$. Note that the options of $(x, y + b)$ are $S + b$ and (x, y).

If $a = 0$, then $\mathcal{SG}(S)$ does not contain 0, so by induction $\mathcal{SG}(S+b)$ does not contain 1. Hence $\mathcal{SG}(x, y+b) = 1$.

If $a = 1$, then $\mathcal{SG}(S)$ does not contain 1, so by induction $\mathcal{SG}(S+b)$ does not contain 0. Hence $\mathcal{SG}(x, y+b) = 0$.

If $a \in \{2, 3\}$, then by induction, $\mathcal{SG}(x, y - b)$, if it exists, is in $\{2, 3\}$, so $\mathcal{SG}(S) = \{0, 1\}$. Then by induction, $\mathcal{SG}(S + b) = \{0, 1\}$. Hence $\mathcal{SG}(x, y + b) = \text{mex}\{0, 1, a\} = a \oplus 1$. ☐

Lemma 1.6 follows immediately. So from now on we focus on the first component of the period $L(b; x_1; y_1; x_2, y_2)$.

Definition 2.2. An array $\{\mathcal{SG}(x)\}_{x\in\mathbb{N}^n}$ is *periodic in* u for some $u \in \mathbb{N}^n$ if $\mathcal{SG}(x) = \mathcal{SG}(x + u)$ for all $x \in \mathbb{N}^n$

In other words, any Lengyel transfer game $L(b; x_1, y_1; x_2, y_2)$ is periodic in $(0, 2b)$. If it is also periodic in $(p, 0)$, we either say that such a game has *horizontal periodicity* (with period p), or simply say that it is periodic with period p.

The next three results are almost immediate consequences of nim-periodicity.

Corollary 2.3. *If* $y_1 \equiv y_1' \bmod 2b$ *and* $y_2 \equiv y_2' \bmod 2b$, *then* $L(b; x_1, y_1; x_2, y_2)$ *and* $L(b; x_1, y_1'; x_2, y_2')$ *have the same* nim*-values.*

Proof. We proceed by induction on the lexicographic order of (x, y). Assume that both games agree about the nim-values of all points less than (x, y). By nim-periodicity, $\mathcal{SG}(x - x_i, y + y_i) = \mathcal{SG}(x - x_i, y + y_i')$, $1 \leq i \leq 2$. Thus both games agree about the value of $\mathcal{SG}(x, y)$. ☐

And so we restrict our analysis to the case $0 \leq y_1, y_2 < 2b$.

Corollary 2.4. *In* $L(b; x_1, y_1; x_2, y_2)$, *the rows numbered* $0, \ldots, b - 1 \bmod 2b$ *cannot contain any* 3*s. The other rows cannot contain any* 2*s*

Proof. Each of the positions in the first b rows have at most 2 options. So these rows only contain the nim-values 0, 1, and 2. The rest follows by induction and nim-periodicity. ☐

nim-periodicity also allows us to show that all Lengyel transfer games are eventually periodic.

Lemma 2.5. *Let* $L(b; x_1, y_1; x_2, y_2)$ *be a Lengyel transfer game. Then it is eventually horizontally periodic.*

Proof. Our first step is to show that $L(b; x_1, y_1; x_2, y_2)$ has eventual row periodicity, as defined in [2].

Let $M = \max\{x_1, x_2\}$. Then nim-periodicity implies that each block of M columns is a function of the first $2b$ rows of the previous M columns. So by the pigeonhole principle, these must eventually repeat. ☐

We may also restrict our analysis to the cases where the horizontal/vertical components are relatively prime.

Lemma 2.6. *For any vector game defined by* $(x_i, y_i)_{i\in I}$, *and any positive integers* m, n, *the array of nim values of the game* $(m \cdot x_i, n \cdot y_i)_{i\in I}$ *is given by that of* $(x_i, y_i)_{i\in I}$, *but with each entry replaced by an* $m \times n$ *block copy of itself. In particular, if the period of* $(x_i, y_i)_{i\in I}$ *is* (p, q) *with* $p, q > 1$, *then the period of* $(m \cdot x_i, n \cdot y_i)_{i\in I}$ *is* (mp, nq). *In the case* $p = 1$ *or* $q = 1$, *that part of the period remains the same.*

Note that this lemma is easily generalizable to more than 2 dimensions, but we state it this way for clarity.

Proof. Essentially [3, Proposition 2.2]. ☐

Corollary 2.7. *If the period of* $L(b;x_1,y_1;x_2,y_2)$ *is* (p,q), *with* $p,q>1$, *then the period of* $L(nb;mx_1,ny_1;mx_2,ny_2)$ *is* (mp,nq). *If either of* p *or* q *is* 1, *then that component of the period remains* 1.

Hence we may restrict our analysis to the case when $\gcd\{x_1,x_2\}=\gcd\{b,y_1,y_2\}=1$. Note also that

$$g(nb,mx_1,ny_1,mx_2,ny_2)=\frac{2nb(mx_1+mx_2)}{\gcd\{2nb,ny_1+ny_2\}}=\frac{(nm)2b(x_1+x_2)}{(n)\gcd\{2b,y_1+y_2\}}=m\cdot g(b,x_1,y_1,x_2,y_2),$$

so if Theorem 1.7 holds for $L(b;x_1,y_1;x_2,y_2)$, then it holds for $L(nb;mx_1,ny_1;mx_2,ny_2)$.

We are now ready to prove a degenerate version of Theorem 1.5, which will also serve as a step in its proof. Let $L(b;x_1,y_1)$ denote the vector game defined by $\{(0,-b),(-x_1,y_1)\}$. Figure 3 shows the initial periods for the games $L(2;1,1)$ and $L(3;1,1)$.

$$\begin{array}{cccc|cccc}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\\
\hdashline
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1\\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}
\qquad
\begin{array}{ccc|ccc}
0 & 1 & 0 & 0 & 1 & 0\\
0 & 1 & 1 & 0 & 1 & 1\\
0 & 0 & 1 & 0 & 0 & 1\\
\hdashline
1 & 0 & 1 & 1 & 0 & 1\\
1 & 0 & 0 & 1 & 0 & 0\\
1 & 1 & 0 & 1 & 1 & 0
\end{array}$$

FIGURE 3. Two initial periods of the arrays of NIM-values for $L(2;1,1)$ and $L(3;1,1)$

Theorem 2.8 (Zeroeth Main Theorem). *If* y_1 *is an odd multiple of* b, *then the game* $L(b;x_1,y_1)$ *has period* $(1,2b)$. *Otherwise, it has period* $\left(\frac{2bx_1}{\gcd\{2b,y_1+b\}},2b\right)$.

Note that $\frac{2bx_1}{\gcd\{2b,y_1+b\}}=g(b,x_1,y_1,0,-b)$, so this is also a kind of degenerate case of Theorem 1.7.

Proof. In light of Corollary 2.7, we may restrict our attention to the case $x_1=1$.

The first $2b$ entries in the first column are b 0s followed by b 1s. After this, each column is a repeat of the previous column, but shifted up by y_1 positions and $\oplus$1-ed. In other words, it is as if each column is an upward shift of the previous column by y_1+b units. Hence it takes $\frac{2b}{\gcd\{2b,y_1+b\}}$ of these shifts for a column to repeat.

Note that $\gcd\{2b,y_1+b\}=2b$ precisely when y_1 is an odd multiple of b. So in this case $\frac{2b}{\gcd\{2b,y_1+b\}}=1$. $\square$

Corollary 2.9. *The period of* $L(b;1,1)$ *is* $(2b,2b)$ *if* b *is even, and* $(b,2b)$ *if* b *is odd.*

2.1. **Generalized Nim-Periodicity.**

One-dimensional vector subtraction games satisfy the Ferguson pairing property, which states that positions of value 0 and 1 are paired by the smallest move.

Proposition 2.10. [1] *Given* $n_1<n_2<\ldots<n_k$, *then in* SUBTRACT$(n_1,n_2,\ldots,n_k)$, $\mathcal{SG}(x)=1$ *if and only if* $\mathcal{SG}(x-n_1)=0$.

In $L(b;x_1,y_1;x_2,y_2)$, we can hear the vector $(0,-b)$ as the 'smallest' move in the lexicographic order. If we do, then NIM-periodicity rhymes with the Ferguson pairing property.

The idea behind NIM-periodicity applies to a larger class of games, so we also define it in a broader context. To see how the general version is applied to Lengyel transfer games, we first redefine vector games in terms of partial functions. In particular, if a vector game is defined by $S=\{u_i\}_{i\in I}\subseteq\mathbb{Z}^n$, then we can instead think of each u_i as a partial function f_i on $\mathbb{N}^n$, which adds u_i when possible. Then the options of any position v are precisely $\{f_i(v)\mid i\in I \text{ and } v\in\operatorname{dom}f_i\}$. So in $L(b;x_1,y_1;x_2,y_2)$, the partial functions are $f_1(x,y)=(x-x_1,y+y_1)$, $f_2(x,y)=(x-x_2,y+y_2)$, $f_3(x,y)=(x,y-b)$, with, for example, $\operatorname{dom}f_3=\{(x,y)\mid y\geq b\}$.

Definition 2.11. Suppose R is an impartial ruleset where the options of any game position $x\in R$ are given by the partial functions $\{f_i\}_{i\in I}$. We say R is NIM-*periodic in* f_i if for all $x\in\operatorname{dom}f_i$, $\mathcal{SG}(f_i(x))=\mathcal{SG}(x)\oplus 1$. In particular, $\mathcal{SG}(f_i(f_i(x)))=\mathcal{SG}(x)$.

Lemma 2.12 (Generalized NIM-Periodicity). *Suppose* R *is an impartial ruleset where the options of any game position* $x\in R$ *are given by the partial functions* $\{f_i\}_{i\in I}$. *Suppose there is* $i\in I$ *with the following properties:*

(i) for all $j \in I$, $f_i \circ f_j = f_j \circ f_i$ when defined
(ii) for all $x \in R$, if $f_i(x)$ exists, then for all $j \in I, j \neq i$,

$$f_j(x) \text{ exists } \iff f_j(f_i(x)) \text{ exists } \iff f_i(f_j(x)) \text{ exists}$$

then R is NIM*-periodic in f_i*

Note that all vector games satisfy the first condition, and that for $L(b; x_1, y_1; x_2, y_2)$, the second condition is satisfied by $f_i(x, y) = (x, y - b)$, showing that, as we have seen, Lengyel transfer games are NIM-periodic in $(0, -b)$.

Every part of condition (ii) is necessary. For example, consider the game $\{(0, -1), (-2, 1)\}$, which is NIM-periodic in $(0, -1)$. Suppose we say that a player cannot do the move $(0, -1)$ if the y-value is a multiple of 3. This no longer satisfies

$$f_j(f_i(x)) \text{ exists} \iff f_i(f_j(x)) \text{ exists,}$$

and we lose NIM-periodicity.

Proof. Suppose R and f_i are as stated. Consider $x \in R$ so that $f_i(x)$ exists. The conditions of the lemma imply that, for all x, $\{\mathcal{SG}(f_j(f_i(x)))\}_{j \neq i} = \{\mathcal{SG}(f_i(f_j(x)))\}_{j \neq i}$. We proceed by structural induction on $x \in \operatorname{dom} f_i$. Fix such an x, and let $a = \mathcal{SG}(f_i(x))$. By induction,

(A) for all j, if $f_i(f_j(x))$ exists, then $\mathcal{SG}(f_i(f_j(x))) = \mathcal{SG}(f_j(x)) \oplus 1$; and
(B) $\mathcal{SG}(f_i(f_i(x)))$, if it exists, is $a \oplus 1$.

If a is even, $a \oplus 1 = a + 1$, so by (B), $\{\mathcal{SG}(f_j(f_i(x)))\}_{j \neq i} = \{\mathcal{SG}(f_i(f_j(x)))\}_{j \neq i}$ contains $\{0, 1, \ldots, a - 1\}$ and not a. Then by (A), $\{\mathcal{SG}(f_j(x))\}_{j \in I}$ contains $\{0 \oplus 1, 1 \oplus 1, \ldots, (a - 1) \oplus 1\} = \{0, 1, \ldots, a - 1\}$ and not $a \oplus 1 = a + 1$. But then $\mathcal{SG}(f_i(x)) = a$, so $\mathcal{SG}(x) = a + 1 = a \oplus 1$.

If a is odd, $a \oplus 1 = a - 1$. Whether or not $\mathcal{SG}(f_i(f_i(x)))$ exists, $\{\mathcal{SG}(f_j(f_i(x)))\}_{j \neq i} = \{\mathcal{SG}(f_i(f_j(x)))\}_{j \neq i}$ contains $\{0, 1, \ldots, a - 2\}$ and not a. Then by (A), $\{\mathcal{SG}(f_j(x))\}_{j \in I}$ contains $\{0 \oplus 1, 1 \oplus 1, \ldots, (a - 2) \oplus 1\} = \{0, 1, \ldots, a - 2\}$ and not $a \oplus 1 = a - 1$. Also, $\mathcal{SG}(f_i(x)) = a$, so $\mathcal{SG}(x) = a - 1 = a \oplus 1$. □

Example 2.13. The vector game defined by

$$\{(-3, 0, 5), (-2, 1, 0), (-1, 1, 1), (0, -3, 0), (0, 0, -4)\}$$

is NIM-periodic in $(0, -3, 0)$ and in $(0, 0, -4)$.

It is a direct consequence of the Sprague-Grundy theory that if every position of an impartial game has at most k options, then the the maximum NIM-value is at most k. In the proof of Theorem 2.8, we see a two-move game whose maximum NIM-value is only 1. In fact, this is not an accident, but a direct consequence of NIM-periodicity. That is, the positions in the first b rows each only have one option, so these values can only be 0 or 1. Then by induction and NIM-periodicity, the remaining rows can only have values from $\{0, 1\}$ or $\{0 \oplus 1, 1 \oplus 1\} = \{0, 1\}$. More generally, we have the following.

Lemma 2.14. *Suppose R is an impartial ruleset where the options of any game position $x \in R$ are given by the partial functions $\{f_i\}_{1 \leq i \leq k}$. Suppose further that R is* NIM*-periodic in f_i for all $1 \leq i \leq l$. Then the maximum* NIM*-value for R is*

$$\max\{k - l, k - l \oplus 1\}$$

So for example, in Example 2.13, $k = 5$ and $l = 2$, so the maximum NIM-value is 3.

Proof. Let $S = \{x \in R \mid x \notin \operatorname{dom} f_i, \text{ for all } 1 \leq i \leq l\}$. Then the maximum NIM-value of any element of S is $k - l$. Then by NIM-periodicity, the maximum NIM-value of any element of R is $\max\{k - l, k - l \oplus 1\}$. □

3. TWO-MOVE GAMES, DIAGONAL PERIODICITY, AND 2-COLUMNS

In [2], Larsson, Saha, and Yokoo made an extensive study of two- and three-move vector subtraction games. In particular, they proved the following.

Lemma 3.1. [2] *In the two-move vector subtraction game defined by the vectors $\{-u, -v\}$, for any position x, the outcome class of x is the same as that of $x + u + v$.*

Since the maximum NIM-value of a two-move impartial game is 2, the following refinement might not seem significant. However, our analysis of Lengyel transfer games will rely on it. Furthermore, NIM-periodicity already shows that it is useful to group our NIM-values as $\{0, 1 \mid 2, 3\}$ rather than as $\{0 \mid 1, 2, 3\}$.

Lemma 3.2 (Two-move periodicity)**.** *The two-move vector subtraction game defined by the vectors* $\{-u, -v\}$ *is periodic in* $u + v$*.*

Proof. In a two move game, the only possible NIM-values are 0, 1, and 2. Use Figure 4 to aid visualization.

If $\mathcal{SG}(x) = 0$, then $\mathcal{SG}(x + v)$ and $\mathcal{SG}(x + u)$ are both positive. Hence $\mathcal{SG}(x + u + v) = 0$.

If $\mathcal{SG}(x) = 1$, then $\mathcal{SG}(x + u), \mathcal{SG}(x + v) \in \{0, 2\}$. We claim that at least one of these values is 0. Indeed if $\mathcal{SG}(x + u) = \mathcal{SG}(x + v) = 2$, then $\mathcal{SG}(x + u - v) = \mathcal{SG}(x + v - u) = 0$. However, this implies that neither of $\mathcal{SG}(x - u)$ or $\mathcal{SG}(x - v)$, if they exist, are zero, contradicting that $\mathcal{SG}(x) = 1$.

If $\mathcal{SG}(x) = 2$, then $\{\mathcal{SG}(x - u), \mathcal{SG}(x - v)\} = \{0, 1\}$. So by the previous two cases, $\{\mathcal{SG}(x + v), \mathcal{SG}(x + u)\} = \{0, 1\}$. Thus $\mathcal{SG}(x + u + v) = 2$. ☐

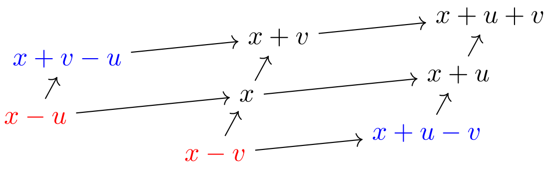


FIGURE 4. Proof of Lemma 3.2

Note that the 0 case follows without looking at any options of x, the 1 case requires looking only at the immediate options of x, and the 2 case follows from the other two cases. In other words, games may exhibit this behavior even if they are only 'locally' a 2-move game.

Due to NIM-periodicity, many Lengyel transfer games $L(b; x_1, y_1; x_2, y_2)$ exhibit a similar periodicity as the two-move vector game obtained if the move $(-b, 0)$ were not available.

Definition 3.3. For any $L(b; x_1, y_1; x_2, y_2)$, let $\mathcal{SG}^*(x, y) = \begin{cases} \mathcal{SG}(x, y) & \mathcal{SG}(x, y) = 0, 1, 2 \\ 2 & \mathcal{SG}(x, y) = 3 \end{cases}$

Definition 3.4. Consider the array of NIM-values of a particular $L(b; x_1, y_1; x_2, y_2)$. We say it has *diagonal periodicity* if, for all (x, y) with $y \geq y_1 + y_2$, we have $\mathcal{SG}^*(x, y) = \mathcal{SG}^*(x + x_1 + x_2, y - y_1 - y_2)$. Similarly, we say it has diagonal periodicity after some x_0 if the condition holds for all $x \geq x_0$.

In other words, $L(b; x_1, y_1; x_2, y_2)$ is diagonally periodic if applying both of the moves $(-x_1, y_1)$ and $(-x_2, y_2)$ does not change the NIM-value, except possibly switching 2s and 3s. In this case, Corollary 2.4 shows us when these switches must occur: between alternating blocks of b rows. Refer to Figure 2, where the 'diagonal direction' in question is $(4, -1)$. Note that we are now mentally partitioning NIM-values as $\{0 \mid 1 \mid 2, 3\}$. We will prove a much stronger version of the following in Section 7, but for now, we lay down some necessary groundwork.

Proposition 3.5. *If* $x_1 \geq 2x_2$ *or* $x_2 \geq 2x_1$*, then* $L(b; x_1, y_1; x_2, y_2)$ *is diagonally periodic.*

In order to prove this proposition, we need another structural fact about Lengyel transfer games. That is, they follow a pattern where the columns alternate between blocks without 2s and 3s, and blocks which might contain them.

Definition 3.6. Consider the array of NIM-values of $L(b; x_1, y_1; x_2, y_2)$. A column is called a 2*-column* if it contains some 2s (equivalently, some 3s).

Lemma 3.7 (The 2-block Rule)**.** *Suppose* $x_2 \leq x_1$*. Then the columns of the* NIM*-array of* $L(b; x_1, y_1; x_2, y_2)$ *numbered* $0, 1, \ldots, x_1 - 1 \bmod (x_1 + x_2)$ *are not* 2*-columns.*

In this case, we call the columns numbered $x_1, x_1 + 1, \ldots, x_1 + x_2 - 1 \bmod (x_1 + x_2)$ *potential* 2*-columns.* We then call a consecutive block of x_2 potential 2-columns a 2*-block.*

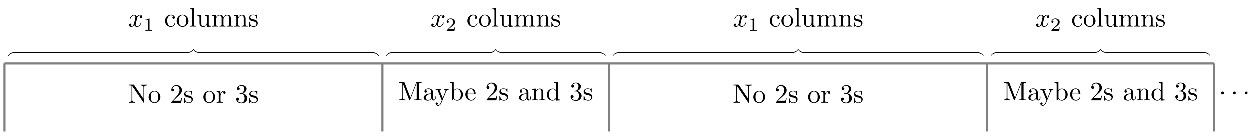


FIGURE 5. The 2-block Rule (Lemma 3.7)

Proof. By NIM-periodicity, we may restrict our attention to the first b rows.

The positions in the first x_1 columns each then have at most 1 option, so the result follows here.

Now we show the result by induction for $x \geq x_1+x_2$ and $x = 0, 1, \ldots, x_1-1 \bmod (x_1+x_2)$. Indeed let x be the smallest such value so that $\mathcal{SG}(x, y) = 2$ for some y. This means $\{\mathcal{SG}(x-x_1, y+y_1), \mathcal{SG}(x-x_2, y+y_2)\} = \{0, 1\}$. By the choice of x, $\mathcal{SG}(x - x_1 - x_2, y + y_1 + y_2) \in \{0, 1\}$, but $(x - x_1 - x_2, y + y_1 + y_2)$ is an option of both $(x - x_1, y + y_1)$ and $(x - x_2, y + y_2)$, a contradiction. ☐

Next we show that we have diagonal periodicity once a particular configuration of 2s and 3s are absent.

Definition 3.8. Suppose $\mathcal{SG}^*(x, y) = \mathcal{SG}^*(x', y') = 2$, where $(x - x', y - y') = (x_1 - x_2, y_2 - y_1)$. We call (x, y) and (x', y') a *bad pair of* 2*s*.

In this case, (x, y) and (x', y') share the option $(x - x_1, y + y_1) = (x' - x_2, y' + y_2)$. In other words, (x, y) and (x', y') are the the red positions in Figure 6.

Lemma 3.9. *If $L(b; x_1, y_1; x_2, y_2)$ has no bad pairs of* 2*s for all $x \geq x_0$, then it has diagonal periodicity after $x_0 + \max\{x_1, x_2\}$.*

Proof. Consider (x, y) so that $0 \leq y-y_1-y_2 < b$. In this case, $(x+x_1+x_2, y-y_1-y_2)$ has only two options, so we may apply the methods of the proof of Lemma 3.2. The proof of the remaining cases follows by NIM-periodicity. See Figure 6.

Suppose $x \geq x_0$. If $\mathcal{SG}(x, y) = 0$, then $\mathcal{SG}(x + x_1, y - y_1)$ and $\mathcal{SG}(x + x_2, y - y_2)$ are both positive. Hence $\mathcal{SG}(x + x_1 + x_2, y - y_1 - y_2) = 0$. If $\mathcal{SG}(x, y) = 1$, then $\mathcal{SG}(x + x_1, y - y_1), \mathcal{SG}(x + x_2, y - y_2) \in \{0, 2, 3\}$. If neither of these were 0, they would be a bad pair of 2s. Hence $\mathcal{SG}(x + x_1 + x_2, y - y_1 - y_2) = 1$.

Now suppose $x \geq x_0 + \max\{x_1, x_2\}$. If $\mathcal{SG}(x, y) \in \{2, 3\}$, then $\mathcal{SG}(x, y - b)$, if it exists, is either 2 or 3, by NIM-periodicity. Hence $\{\mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - x_2, y + y_2)\} = \{0, 1\}$. Then by the previous cases, $\{\mathcal{SG}(x + x_1, y - y_1), \mathcal{SG}(x + x_2, y - y_2)\} = \{0, 1\}$, and so $\mathcal{SG}(x + x_1 + x_2, y - y_1 - y_2) \in \{2, 3\}$ ☐

Proof of Proposition 3.5. Suppose $x_2 \geq 2x_1$. We show that there are no bad pairs of 2s. Indeed if (x, y) and (x', y') is a bad pair of 2s, then $x-x' = x_2-x_1$ and so $x_1 \leq x-x' \leq x_2$. In other words, $x-x' \geq x_1 \bmod (x_1+x_2)$, violating Lemma 3.7. ☐

$$\begin{tikzcd}
 & & (x+x_1+x_2, y-y_1-y_2) \\
(x+x_2, y-y_2) \arrow[urr] & & \\
 & & (x+x_1, y-y_1) \arrow[uu] \\
(x, y) \arrow[uu] \arrow[urr] & &
\end{tikzcd}$$

FIGURE 6. Proof of Proposition 3.9

Finally, we show that, as we move from one 2-block to the next, there are no new 2s or 3s, and that the 2s and 3s which remain follow diagonal periodicity.

Lemma 3.10. *If $\mathcal{SG}^*(x + x_1 + x_2, y - y_1 - y_2) = 2$, then $\mathcal{SG}^*(x, y) = 2$.*

Proof. Following Figure 6, let $\mathcal{SG}^*(x + x_1 + x_2, y - y_1 - y_2) = 2$. Then, by NIM-periodicity, $\mathcal{SG}^*(x + x_1 + x_2, y - y_1 - y_2 - b)$, if it exists, is 2. Hence $\{\mathcal{SG}(x + x_1, y - y_1), \mathcal{SG}(x + x_2, y - y_2)\} = \{0, 1\}$. Thus $\mathcal{SG}^*(x, y) = 2$. ☐

4. Proof of Theorem 1.5: The Period of $L(b; x_1, 0; 1, 1)$

Proposition 4.1. *Assume that* $(b, x_1) \neq (1, 1)$. *The game* $L(b; x_1, 0; 1, 1)$ *has diagonal periodicity.*

Note that Figure 1 shows that the conclusion fails if $(b, x_1) = (1, 1)$.

Proof. Proposition 3.5 implies the result if $x_1 \geq 2$.

Assume $x_1 = 1$ and $b \geq 2$. Lemma 3.7 says that exactly the odd numbered columns are potential 2-columns. Then by inspection, we see that the first 2-block does not contain any bad pairs of 2s (See Figure 7, for example), so by Lemma 3.10, there are not any at all. Then Lemma 3.9 implies the result. ☐

0	1	0	1	0	2	1	0	1	0	1	2
0	1	0	2	1	0	1	0	1	2	0	1
0	2	1	0	1	0	1	2	0	1	0	1
1	0	1	0	1	3	0	1	0	1	0	3
1	0	1	3	0	1	0	1	0	3	1	0
1	3	0	1	0	1	0	3	1	0	1	0

Figure 7. A period of the nim-values for $L(3; 1, 0; 1, 1)$

Next Lemma 3.7 says that the first x_1 columns are not 2-columns. The cases (b) and (c) of Theorem 1.5 are determined by whether the $(x_1 + 1)$th is a 2-column.

Lemma 4.2. *The* $(x_1 + 1)$*th column of the game* $L(b; x_1, 0; 1, 1)$ *is a* 2*-column, except when* b *is even and* x_1 *is an odd multiple of* b.

Proof. By nim-periodicity, we just need to check whether any of the first b entries of this column have nim-value 2. Note that each of these positions only have 2 options.

The first x_1 columns of $L(b; x_1, 0; 1, 1)$ are identical to that of $L(b; 1, 1)$. In particular, the first b entries of the first column are all 0.

Now consider the $(x_1 + 1)$th column of $L(b; 1, 1)$. Observing Figure 3, we see that this column contains some 0s in the first b entries, except in the case that b is even and x_1 is an odd multiple of b.

In the first case, consider one of these 0 entries, (x_1, y). Then $\mathcal{SG}(x_1 - 1, y + 1)$ must be 1, and $\mathcal{SG}(0, y) = 0$. Hence, in the game $L(b; x_1, 0; 1, 1)$, we have $\mathcal{SG}(x, y) = 2$.

In the latter case, when $\mathcal{SG}(x_1, y) = 1$ for all $0 \leq y \leq b - 1$, we must have $\mathcal{SG}(x_1 - 1, y + 1) = 0$ for all $0 \leq y \leq b - 1$. Since we also have $\mathcal{SG}(0, y) = 0$ for all $0 \leq y \leq b - 1$, then in the game $L(b; x_1, 0; 1, 1)$, we have $\mathcal{SG}(x_1, y) = 1$ for all $0 \leq y \leq b - 1$. ☐

Proof of Theorem 1.5. Again, the proof of part (a) is essentially Figure 1. So assume $(b, x_1) \neq (1, 1)$.

Suppose first that it is not the case that b is even and x_1 is an odd multiple of b. By Lemma 4.2, the $(x_1 + 1)$th column is a 2 column. Next, by Proposition 4.1, the first $x_1 + 1$ columns repeat with an upward shift of 1 unit. We then know that it takes $2b$ such shifts to repeat, since the first column of the period is precisely b 0s followed by b 1s. Hence the horizontal component of the period is $2b(x_1 + 1)$

In the case that b is even and x_1 is an odd multiple of b, Lemma 4.2 states that the $(x_1 + 1)$th column does not contain any 2s or 3s. Then Proposition 4.1 implies that no later columns do either. The only way for this to happen is if, for any such position, all options have the same nim-value. In other words, the values of this game are same as that of $L(b; 1, 1)$, which, by Corollary 2.9, has a period whose horizontal component is $2b$ when b is even. ☐

5. Long preperiods

In this section, we examine a set of examples of $L(b; x_1, 0; x_2, y_2)$ with arbitrarily long preperiod. One set has period 2, showing that the preperiod can be arbitrarily longer than the period. The other set has period $g(b, x_1, 0, x_2, y_2)$, showing that we have arbitrarily long preperiod when the period is maximal.

Proposition 5.1. *The following games have the following horizontal preperiods and periods:*

		preperiod	*period*
i	$L(b;1,0;1,b-1)$, *b odd*	$b-1$	2
ii	$L(b;1,0;1,b-1)$, *b even*	$b-2$	$4b$
iii	$L(b;1,0;1,b+1)$, *b odd*	$b-1$	2
iv	$L(b;1,0;1,b+1)$, *b even*	$b-2$	$4b$

Proof. Figure 8 shows the preperiod and the first two columns of the period for each case of $b = 7$ and $b = 8$. After careful inspection, one finds that the pattern persists for other values of b. □

We have highlighted the bad pairs of 2s in Figure 8 so the reader can observe the exact manner in which they disappear over the course of the preperiod.

6. Simplifying assumptions

We work toward establishing a coherent conjecture about the horizontal period of $L(b; x_1, 0; x_2, y_2)$. To this end, we establish some simplifying assumptions. Note that the results in this and the next section do apply to the more general case where y_1 is not necessarily 0.

- If $y_2 = 0$, then $L(b; x_1, 0; x_2, y_2)$ is isomorphic to the one-dimensional, two-move subtraction game defined by x_1 and x_2. In this case, apply Lemma 3.1 to the first $x_1 + x_2$ values.
- If $y_1 \geq 2b$ or $y_2 \geq 2b$, then apply Corollary 2.3.
- If $\gcd\{x_1, x_2\} > 1$ or if $\gcd\{b, y_1, y_2\} > 1$, then apply Corollary 2.7.

Our last observation is that sometimes one of the vectors defining a game is redundant, and we can thus can be eliminated. Then the NIM-values can be computed via Theorem 2.8.

Lemma 6.1 (Vector Elimination)**.** *Consider $L(b; x_1, y_1; x_2, y_2)$. Suppose that either*

- *There is an odd k so that $x_2 = kx_1$ and $y_2 = ky_1 \bmod 2b$; or*
- *There is an even k so that $x_2 = kx_1$ and $y_2 = ky_1 + b \bmod 2b$*

then $L(b; x_1, y_1; x_2, y_2)$ has the same NIM*-values as $L(b; x_1, y_1)$. In particular, it has no 2s or 3s.*

Proof. In the first case, the result is clear when $x < x_2$. Now consider some (x, y) with $x \geq x_2$, and proceed by induction on x. Suppose further that $y < b$, as the result for $y \geq b$ will then follow by NIM-periodicity.

In $L(b; x_1, y_1)$, since the only NIM-values are 0 and 1, we have that $\mathcal{SG}(p) = \mathcal{SG}(q) \oplus 1$ whenever q is an option of p. Hence

$$\mathcal{SG}(x, y), \mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - 2x_1, y + 2y_1), \ldots, \mathcal{SG}(x - kx_1, y + ky_1)$$

is an alternating 0, 1-sequence in $L(b; x_1, y_1)$. By induction,

$$\mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - 2x_1, y + 2y_1), \ldots, \mathcal{SG}(x - kx_1, y + ky_1)$$

is the same alternating 0, 1-sequence in $L(b; x_1, y_1; x_2, y_2)$. In particular, since k is odd, $\mathcal{SG}(x - x_1, y + y_1) = \mathcal{SG}(x - kx_1, y + ky_1)$. But then in $L(b; x_1, y_1; x_2, y_2)$,

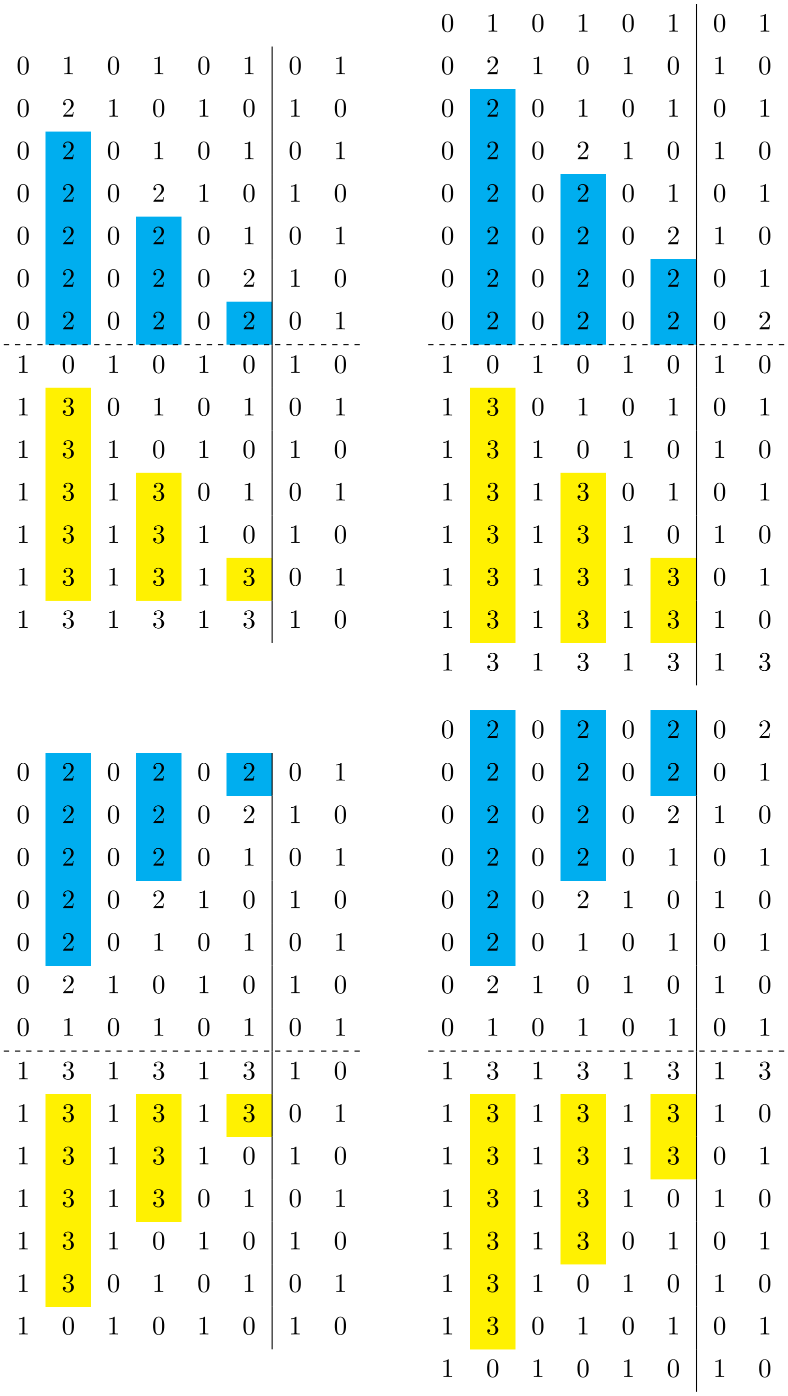
$$\begin{aligned}
\mathcal{SG}(x, y) &= \operatorname{mex}\{\mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - x_2, y + y_2)\} \\
&= \operatorname{mex}\{\mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - kx_1, y + ky_1)\} \\
&= \operatorname{mex}\{\mathcal{SG}(x - x_1, y + y_1)\} \\
&= \mathcal{SG}(x - x_1, y + y_1) \oplus 1
\end{aligned}$$

which is the same value as in $L(b; x_1, y_1)$.

In the second case, the proof is similar, except that since k is even, we have $\mathcal{SG}(x - x_1, y + y_1) = \mathcal{SG}(x - kx_1, y + ky_1) \oplus 1$. Then in $L(b; x_1, y_1; x_2, y_2)$,

$$\begin{aligned}
\mathcal{SG}(x, y) &= \operatorname{mex}\{\mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - x_2, y + y_2)\} \\
&= \operatorname{mex}\{\mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - kx_1, y + ky_1 + b)\} \\
&= \operatorname{mex}\{\mathcal{SG}(x - x_1, y + y_1), \mathcal{SG}(x - kx_1, y + ky_1) \oplus 1\} && \text{by NIM-periodicity} \\
&= \operatorname{mex}\{\mathcal{SG}(x - x_1, y + y_1)\} \\
&= \mathcal{SG}(x - x_1, y + y_1) \oplus 1
\end{aligned}$$

which is the same value as in $L(b; x_1, y_1)$. □



0	1	0	1	0	1	0	1
0	2	1	0	1	0	1	0
0	2	0	1	0	1	0	1
0	2	0	2	1	0	1	0
0	2	0	2	0	1	0	1
0	2	0	2	0	2	1	0
0	2	0	2	0	2	0	1
1	0	1	0	1	0	1	0
1	3	0	1	0	1	0	1
1	3	1	0	1	0	1	0
1	3	1	3	0	1	0	1
1	3	1	3	1	0	1	0
1	3	1	3	1	3	0	1
1	3	1	3	1	3	1	0

0	1	0	1	0	1	0	1
0	2	1	0	1	0	1	0
0	2	0	1	0	1	0	1
0	2	0	2	1	0	1	0
0	2	0	2	0	1	0	1
0	2	0	2	0	2	1	0
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1	0	1	0	1	0	1	0
1	3	0	1	0	1	0	1
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1	3	1	3	0	1	0	1
1	3	1	3	1	0	1	0
1	3	1	3	1	3	0	1
1	3	1	3	1	3	1	0
1	3	1	3	1	3	1	3

0	2	0	2	0	2	0	1
0	2	0	2	0	2	1	0
0	2	0	2	0	1	0	1
0	2	0	2	1	0	1	0
0	2	0	1	0	1	0	1
0	2	1	0	1	0	1	0
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1	3	0	1	0	1	0	1
1	0	1	0	1	0	1	0

0	2	0	2	0	2	0	2
0	2	0	2	0	2	0	1
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1	3	1	3	1	3	1	3
1	3	1	3	1	3	1	0
1	3	1	3	1	3	0	1
1	3	1	3	1	0	1	0
1	3	1	3	0	1	0	1
1	3	1	0	1	0	1	0
1	3	0	1	0	1	0	1
1	0	1	0	1	0	1	0

FIGURE 8. The preperiod of the NIM-values for $L(7;1,0;1,6)$, $L(8;1,0;1,7)$, $L(7;1,0;1,8)$, and $L(8;1,0;1,9)$, together with the first two 2-blocks of the initial period, showing diagonal periodicity. The bad pairs of 2s are highlighted.

Corollary 6.2. *Consider* $L(b;x_1,0;x_2,y_2)$*, where* $y_2 < 2b$*.*

(a) If

- x_2 *is an odd multiple of* x_1 *and* $y_2 = 0$*; or*
- x_2 *is an even multiple of* x_1 *and* $y_2 = b$*,*

then $L(b;x_1,0;x_2,y_2)$ *has the same* NIM*-values as* $L(b;x_1,0)$*.*

(b) If
  - $x_1 = kx_2$*, where* k *is odd, and* ky_2 *is an even multiple of* b*; or*
  - $x_1 = kx_2$*, where* k *is even, and* ky_2 *is an odd multiple of* b*,*

then $L(b; x_1, 0; x_2, y_2)$ *has the same* NIM*-values as* $L(b; x_2, y_2)$*.*

Note that the last case is precisely what happens in Theorem 1.5(b). **For the next two sections, assume the following:**

- $0 \le y_1 < 2b$
- $0 < y_2 < 2b$
- $\gcd\{x_1, x_2\} = 1$
- $\gcd\{b, y_1, y_2\} = 1$ (or $\gcd\{b, y_2\} = 1$ in the case $y_1 = 0$)
- Lemma 6.1 (or Corollary 6.2 in the case $y_1 = 0$) does not apply

7. Eventual diagonal periodicity

In this section, we show that with two exceptions, $L(b; x_1, y_1; x_2, y_2)$ has eventual diagonal periodicity due to the eventual disappearance of bad pairs of 2s. In one of the exceptional cases, we have diagonal periodicity anyway. In the remaining case, we do not have diagonal periodicity, but the chart of NIM-values is trivial.

By a *chain of bad pairs of* 2*s*, we mean a sequence $(x, y), (x + m, y + n), (x + 2m, y + 2n), \ldots$ where each adjacent pair is a bad pair of 2s.

Lemma 7.1. *Suppose* $\mathcal{SG}^*(x, y) = \mathcal{SG}^*(x + x_1 + x_2, y - y_1 - y_2) = 2$*, and that* (x, y) *is not part of two bad pairs of* 2*s. Then* $\mathcal{SG}(x - x_1, y + y_1) = \mathcal{SG}(x + x_2, y - y_2)$ *and* $\mathcal{SG}(x - x_2, y + y_2) = \mathcal{SG}(x + x_1, y - y_1)$*.*

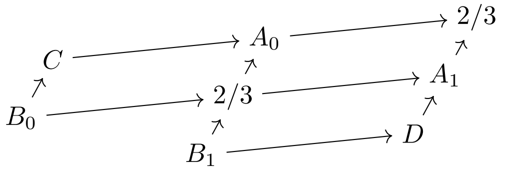


FIGURE 9. Proof of Lemma 7.1. The arrows represent the vectors $(x_1, -y_1)$ and $(x_2, -y_2)$.

Proof. Observe Figure 9, where (x, y) and $(x + x_1 + x_2, y - y_1 - y_2)$ are the positions marked 2/3, and either $\mathcal{SG}^*(C) \ne 2$ or $\mathcal{SG}^*(D) \ne 2$. Suppose $\mathcal{SG}^*(C) \ne 2$. By NIM-periodicity, $(x, y - b)$ and $(x + x_1 + x_2, y - y_1 - y_2 - b)$, if they exist, also each have value 2 or 3. Thus $\{A_0, A_1\} = \{B_0, B_1\} = \{0, 1\}$. Without loss of generality, $A_0 = 0$ and $A_1 = 1$. Then $\mathcal{C} = 1$, so $B_0 = 0$ and $B_1 = 1$. ☐

Proposition 7.2. *Suppose* $y_1 + y_2 \ne 2b$*. Suppose further that either (i)* $x_1 \ne x_2$*, or (ii)* $x_1 = x_2 = 1$ *and* $|y_1 - y_2| \ne b$*. Then any bad pairs of* 2*s in* $L(b; x_1, y_1; x_2, y_2)$ *eventually disappear.*

Proof. Assume toward a contradiction that the bad pairs of 2s do not eventually disappear.

Claim. *For any* k*, we can find a chain of at least* k *bad pairs of* 2*s.*

proof of claim. Suppose not. Then by Lemma 3.10, there is some column x_0, after which all chains of bad pairs of 2s are of the minimal length, say $k_0 > 1$.

By Lemma 7.1, along with the fact that $y_1 + y_2 \ne 2b$, we can select the chain far enough to the right, so that the position (x, y) (corresponding to A_0 or A_1 in Figure 9) satisfies $\mathcal{SG}^*(x - x_1, y + y_1) = \mathcal{SG}^*(x - x_2, y + y_2) = 2$, and either

- $\mathcal{SG}(x, y) = 1$ and $0 \le y < b$; or
- $\mathcal{SG}(x, y) = 0$ and $b \le y < 2b$.

In either case, NIM-periodicity gives us a node (x, y') with $y' < b$, $\mathcal{SG}(x, y') = 1$, and both options of (x, y') having value 2 or 3, a contradiction. ☐

In Case (i), choose a large enough k to violate Lemma 3.7.

In Case (ii), let $(x, y + jm)$, where $m = |y_1 - y_2|$, for $0 \leq j \leq 2b$ be a chain of bad pairs of 2s. Letting $x' = x - x_1, y' = y + y_1$, we obtain a sequence of positions $(x', y' + jm)$, whose values form an alternating sequence of 0s and 1s. Since $m \neq b$, there must be some j so that $y' + jm = 2bl + i$ for some $0 \leq i < b$ and $\mathcal{SG}(x', y' + jm) = 1$. By NIM-periodicity, $\mathcal{SG}(x', i) = \mathcal{SG}(x', y' + jm - 2bl) = 1$. However, since $i < b$, (x', i) only has options from the chain of bad pairs of 2s, a contradiction. ☐

Let's now consider the exceptions to the above proposition.

Proposition 7.3. *Suppose* $y_1 + y_2 = 2b$. *Then* $L(b; x_1, y_1; x_2, y_2)$ *has diagonal periodicity. In particular, it has no preperiod and the horizontal period is* $x_1 + x_2$ $(= g(b, x_1, y_1, x_2, y_2))$.

0	0	0	0	1	2	2	2	2
0	0	0	0	1	2	2	2	2
0	0	0	0	0	0	0	0	2
0	0	0	0	0	2	2	2	1
0	0	0	0	0	2	2	2	2
1	1	1	1	0	3	3	3	3
1	1	1	1	0	3	3	3	3
1	1	1	1	1	1	1	1	3
1	1	1	1	1	3	3	3	0
1	1	1	1	1	3	3	3	3

0	0
0	2
0	0
1	1
1	3
1	1

0	2
0	2
0	2
1	3
1	3
1	3

FIGURE 10. One period each of the NIM-values of $L(5; 5, 4; 7, 3)$, $L(3; 1, 2; 1, 4)$, and $L(3; 1, 2; 1, 5)$

See for example Figure 10 for the values of $L(5; 5, 4; 7, 3)$ and $L(3; 1, 2; 1, 4)$.

Proof. We will show that $\mathcal{SG}^*(x, y) = \mathcal{SG}^*(x + x_1 + x_2, y - 2b)$ in the case $2b \leq y < 3b$. All other cases follow by NIM-periodicity.

Suppose $\mathcal{SG}^*(x, y) = 0$. Then $\mathcal{SG}^*(x + x_1, y - y_1), \mathcal{SG}^*(x + x_2, y - y_2) \neq 0$, so $\mathcal{SG}^*(x + x_1 + x_2, y - 2b) = 0$.

Next, suppose $\mathcal{SG}^*(x, y) = 1$. By Lemma 3.10, $\mathcal{SG}^*(x + x_1 + x_2, y - 2b) \neq 2$, so assume toward a contradiction that $\mathcal{SG}^*(x + x_1 + x_2, y - 2b) = 0$. This means $\mathcal{SG}^*(x + x_1, y - y_1), \mathcal{SG}^*(x + x_2, y - y_2) = 2$. Hence by Lemma 3.10 again, $\mathcal{SG}^*(x - x_1, y + y_1)$ and $\mathcal{SG}^*(x - x_2, y + y_2)$, when they exist, are 2. However, by NIM-periodicity, this means that $\mathcal{SG}^*(x - x_1, y + y_1 - 2b)$ and $\mathcal{SG}^*(x - x_2, y + y_2 - 2b)$, when they exist, are also 2. But these are the only possible options of $(x, y - 2b)$, contradicting that $\mathcal{SG}^*(x, y - 2b) = \mathcal{SG}^*(x, y) = 1$.

Finally, suppose $\mathcal{SG}^*(x, y) = 2$. Then since $\mathcal{SG}^*(x, y - b) = 2$, we must have $\{\mathcal{SG}^*(x - x_1, y + y_1), \mathcal{SG}^*(x - x_2, y + y_2)\} = \{0, 1\}$. Then by the previous cases, $\{\mathcal{SG}^*(x + x_1, y - y_1), \mathcal{SG}^*(x + x_2, y - y_2)\} = \{0, 1\}$. Hence $\mathcal{SG}^*(x + x_1 + x_2, y - 2b) = 2$. ☐

Proposition 7.4. *Suppose* $x_1 = x_2 = 1$ *and* $|y_1 - y_2| = b$. *Then the* NIM*-values are as in the third part of Figure 10. In particular, there is no preperiod, and the horizontal period is* 2.

Proof. See Figure 10 for the case of $L(3; 1, 2; 1, 5)$. All other cases are similar. ☐

Note that we do not have diagonal periodicity, unless we additionally require $y_1 + y_2 = 2b$. This happens precisely when $\{y_1, y_2\} = \{\frac{b}{2}, \frac{3b}{2}\}$. Theorem 1.5(a) is a special case of this. See also Figure 1.

We are now ready to prove our second main theorem.

proof of Theorem 1.7. By the preceding propositions, the only case when $L(b; x_1, y_1; x_2, y_2)$ does not have diagonal periodicity is precisely when $x_1 = x_2$, $|y_1 - y_2| = b \bmod 2b$, and $y_1 + y_2 \neq 2b \bmod 2b$. However, combining Proposition 7.4 with Corollary 2.7 yields a horizontal period of $(x_1 + x_2)$, which is a factor of $g(b, x_1, y_1, x_2, y_2)$.

So suppose $L(b; x_1, y_1; x_2, y_2)$ has diagonal periodicity, and that $2b = k \cdot \gcd\{2b, y_1 + y_2\}$. In other words, $k(y_1 + y_2) = 0 \bmod 2b$. Then by NIM-periodicity and diagonal periodicity,

$$\mathcal{SG}^*(x, y) = \mathcal{SG}^*(x + k(x_1 + x_2), y - k(y_1 + y_2)) = \mathcal{SG}^*(x + k(x_1 + x_2), y).$$

Finally, Corollary 2.4 implies $\mathcal{SG}(x,y) = \mathcal{SG}(x + k(x_1 + x_2), y)$. ☐

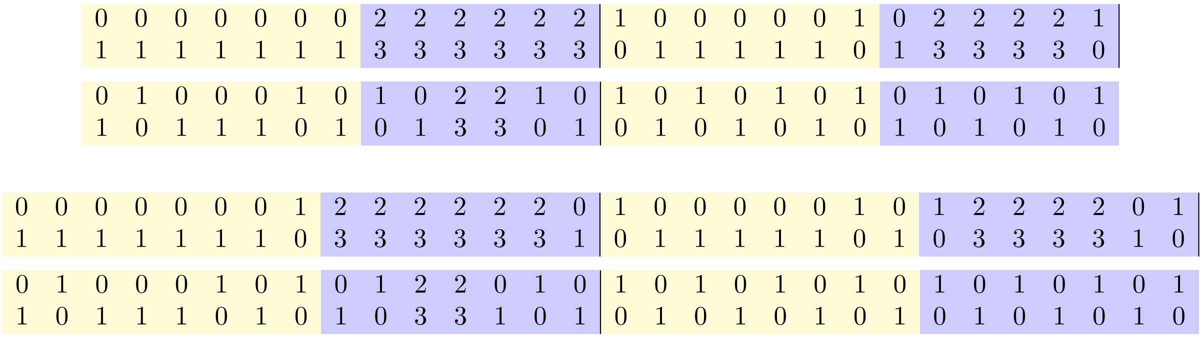
Remark. This proof shows how NIM-periodicity and diagonal periodicity together imply horizontal periodicity, with a (maximum possible) period specified by the same parameters.

For the converse, suppose we assume NIM-periodicity in $(0, -b)$ and a horizontal period of p. Then we obtain periodicity in $(p, -2b)$. In other words, we obtain a kind of diagonal periodicity, but not necessarily in the direction specified by the sum of the moves (x_1, y_1) and (x_2, y_2).

8. THE HORIZONTAL PERIOD OF $L(b; x_1, 0; x_2, y_2)$

8.1. A few more sporadic cases.

Proposition 8.1. *Suppose x_1 is odd and $x_2 = x_1 \pm 1$. Then the horizontal period of $L(1; x_1, 0; x_2, 1)$ is 2.*



0	0	0	0	0	0	0	2	2	2	2	2	2	1	0	0	0	0	0	1	0	2	2	2	2	1
1	1	1	1	1	1	1	3	3	3	3	3	3	0	1	1	1	1	1	0	1	3	3	3	3	0
0	1	0	0	0	1	0	1	0	2	2	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	0	1	1	1	0	1	0	1	3	3	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

0	0	0	0	0	0	0	1	2	2	2	2	2	2	0	1	0	0	0	0	0	1	0	1	2	2	2	2	0	1
1	1	1	1	1	1	1	0	3	3	3	3	3	3	1	0	1	1	1	1	1	0	1	0	3	3	3	3	1	0
0	1	0	0	0	1	0	1	0	1	2	2	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
1	0	1	1	1	0	1	0	1	0	3	3	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

FIGURE 11. The first $2(x_1 + x_2)$ columns for $L(1; 7, 0; 6, 1)$ and $L(1; 7, 0; 8, 1)$

Proof. Figure 11 shows the first $4(7 + 6)$ and $4(8 + 7)$ columns for $L(1; 7, 0; 6, 1)$ and $L(1; 7, 0; 8, 1)$, respectively. We will discuss the former case ($x_2 = x_1 - 1$), and leave the latter to the reader.

The first x_1 columns are all $(0, 1)$, and the next x_2 columns are all $(2, 3)$.

Consider the k^{th} $(x_1 + x_2)$-block after the first one. We describe how it is generated by the previous block. In the first x_1 columns, the initial k and final k columns are flipped. In the last x_2 columns, the initial $k - 1$ and final $k - 1$ columns are flipped, while the k^{th} and $(x_1 + x_2 - k)^{th}$ columns become $(0, 1)$ and $(1, 0)$, respectively.

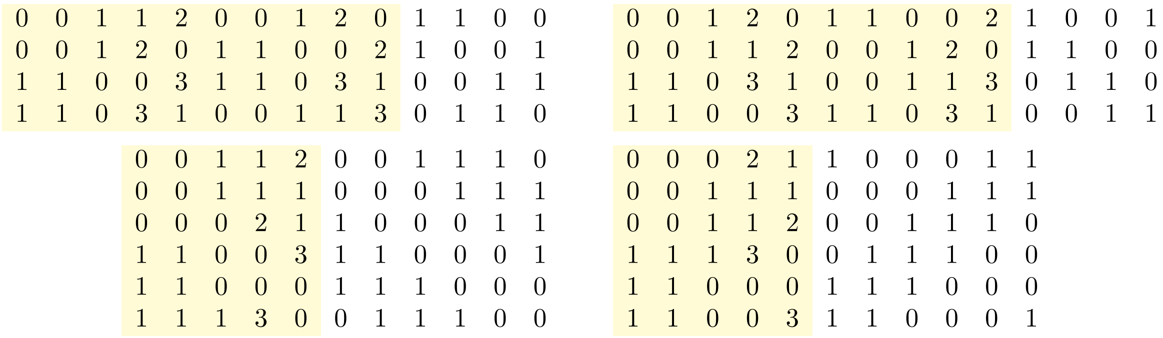
Thus the $\frac{x_1+1}{2}^{th}$ $(x_1 + x_2)$-block consists of alternating sequences of 0s and 1s. Hence the eventual horizontal period is 2. ☐

Proposition 8.2. *The following games have the following horizontal preperiods and periods:*

		preperiod	*period*
i	$L(2; 2, 0; 3, 1)$	10	4
ii	$L(2; 2, 0; 3, 3)$	10	4
iii	$L(3; 3, 0; 2, 1)$	5	6
iv	$L(3; 3, 0; 2, 5)$	5	6

Proof. See Figure 12. In each case, the preperiod is highlighted, and one period follows. ☐

Remark. These sporadic cases arise from a sort of "eventual" vector elimination, where, after the preperiod, the NIM-values line up so that $(x - x_1, y + y_1)$ and $(x - x_2, y + y_2)$ always have the same values. These cases were found via computer models, and we currently have no idea how to predict this phenomenon systematically. In fact, our inability to show when this does *not* happen is our main roadblock to proving Conjecture 8.3.



0	0	1	1	2	0	0	1	2	0	1	1	0	0
0	0	1	2	0	1	1	0	0	2	1	0	0	1
1	1	0	0	3	1	1	0	3	1	0	0	1	1
1	1	0	3	1	0	0	1	1	3	0	1	1	0

0	0	1	2	0	1	1	0	0	2	1	0	0	1
0	0	1	1	2	0	0	1	2	0	1	1	0	0
1	1	0	3	1	0	0	1	1	3	0	1	1	0
1	1	0	0	3	1	1	0	3	1	0	0	1	1

0	0	1	1	2	0	0	1	1	1	0
0	0	1	1	1	0	0	0	1	1	1
0	0	0	2	1	1	0	0	0	1	1
1	1	0	0	3	1	1	0	0	0	1
1	1	0	0	0	1	1	1	0	0	0
1	1	1	3	0	0	1	1	1	0	0

0	0	0	2	1	1	0	0	0	1	1
0	0	1	1	1	0	0	0	1	1	1
0	0	1	1	2	0	0	1	1	1	0
1	1	1	3	0	0	1	1	1	0	0
1	1	0	0	0	1	1	1	0	0	0
1	1	0	0	3	1	1	0	0	0	1

FIGURE 12. The preperiods and initial periods for $L(2;2,0;3,1)$, $L(2;2,0;3,3)$, $L(3;3,0;2,1)$, and $L(3;3,0;2,5)$

8.2. **The main conjecture.** Our computer aided calculations show the following up to $b, x_1, x_2, y_2 \leq 20$. For our Maple code, see the last author's website [4].

Conjecture 8.3. *Suppose* $0 < y_2 < 2b$, $\gcd\{x_1, x_2\} = 1$, *and* $\gcd\{b, y_2\} = 1$. *Suppose further that none of Corollary 6.2 or Propositions 5.1(i),5.1(iii), 7.4, 8.1, or 8.2 apply. Then the horizontal period of* $L(b; x_1, 0; x_2, y_2)$ *is given by*

$$g(b, x_1, 0, x_2, y_2) = \frac{2b(x_1 + x_2)}{\gcd\{2b, y_2\}}.$$

In the general case of $L(b; x_1, y_1; x_2, y_2)$, we have the exact same phenomenon. However, it is more complicated to determine when we have eventual vector elimination, so we do not make a more specific conjecture in this paper.

9. LENGYEL'S OTHER CONJECTURE: MULTIPLE TRANSFER OPTIONS

Lengyel had one other conjecture in [3], and we prove it now.

Definition 9.1. Define the *Lengyel multitransfer game* $L^*(a, b, c)$ to be the vector game defined by

$$\{(-a, 0), (0, -b), (-1, 1), (-2, 2), \ldots, (-c, c)\}$$

Note that we still have NIM-periodicity in $(0, -b)$.

Proposition 9.2. *For* $b \geq 2$, *the horizontal component of the period of* $L^*(b, b, b)$ *is* $b + 2$.

Notice that $L^*(1, 1, 1) = L(1; 1, 0; 1, 1)$.

Proof. We give a proof by diagram in Figure 13 for the cases $b = 6$ and $b = 7$. Note that in each case, there is no preperiod. The remaining cases are similar. ☐

10. OTHER DIRECTIONS. OPEN QUESTIONS.

While we are happy that Theorem 1.5 is a concrete generalization of [3] and how Theorem 1.7 applies to the most general case, we are frustrated at how close the proof to Conjecture 8.3 feels. Nevertheless, we feel that we have proved almost all of it, and laid the groundwork for someone else to complete the job.

Question 1. *Exactly how does the 'eventual vector elimination' work in Conjecture 8.3?*

Perhaps they could also answer (and prove?) the following.

Question 2. *What is the correct conjecture for* $L(b; x_1, y_1; x_2, y_2)$ *when* $y_1 > 0$*?*

Also, section 9 provides fertile ground for exploration. For example:

Question 3. *What is the horizontal period for* $L^*(a, b, c)$ *in general?*

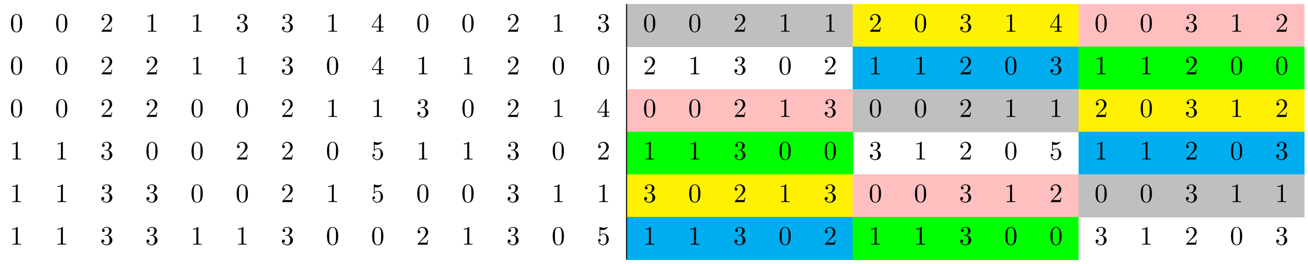
If one relaxes the ciliary muscles as if to view an autostereogram, one sees that this paper is about the interplay between different notions of periodicity: horizontal periodicity, NIM-periodicity, two-move periodicity, and diagonal periodicity. We end with three examples which invite the reader to have as much fun with periodicity as we have had.

0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7
0	0	2	2	4	4	6	6
1	0	3	2	5	4	7	6
1	0	3	2	5	4	7	6
1	0	3	2	5	4	7	6
1	0	3	2	5	4	7	6
1	0	3	2	5	4	7	6
1	1	3	3	5	5	7	7

0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8
0	0	2	2	4	4	6	6	8
1	0	3	2	5	4	7	6	9
1	0	3	2	5	4	7	6	9
1	0	3	2	5	4	7	6	9
1	0	3	2	5	4	7	6	9
1	0	3	2	5	4	7	6	9
1	0	3	2	5	4	7	6	9
1	1	3	3	5	5	7	7	9

FIGURE 13. A period of the NIM-values for $L^*(6,6,6)$ and $L^*(7,7,7)$

Adding transfer options. The vector game $\{(0,-3),(-2,0),(-1,3),(-2,2),(-4,1)\}$ is like a Lengyel transfer game with two extra transfer options. Thus it also has NIM-periodicity and eventual horizontal periodicity. In fact, the (horizontal) preperiod and period are 14 and 15, respectively. Figure 14 shows the preperiod and one period, as highlighted in the figure. If we squint, the values in the period seem to be following a sort of diagonal periodicity in the direction $(5,-4)$. Perhaps we redefine diagonal periodicity using $\mathcal{SG}^*(x,y) = 2$ whenever $\mathcal{SG}(x,y) \geq 2$? But then how do we predict the position of the 4 and 5 in the tenth column of the period? And why is $(5,-4)$ the direction to squint in?



0	0	2	1	1	3	3	1	4	0	0	2	1	3	0	0	2	1	1	2	0	3	1	4	0	0	3	1	2
0	0	2	2	1	1	3	0	4	1	1	2	0	0	2	1	3	0	2	1	1	2	0	3	1	1	2	0	0
0	0	2	2	0	0	2	1	1	3	0	2	1	4	0	0	2	1	3	0	0	2	1	1	2	0	3	1	2
1	1	3	0	0	2	2	0	5	1	1	3	0	2	1	1	3	0	0	3	1	2	0	5	1	1	2	0	3
1	1	3	3	0	0	2	1	5	0	0	3	1	1	3	0	2	1	3	0	0	3	1	2	0	0	3	1	1
1	1	3	3	1	1	3	0	0	2	1	3	0	5	1	1	3	0	2	1	1	3	0	0	3	1	2	0	3

FIGURE 14. The preperiod and one period of the NIM-values of the vector game $\{(0,-3),(-2,0),(-1,3),(-2,2),(-4,1)\}$

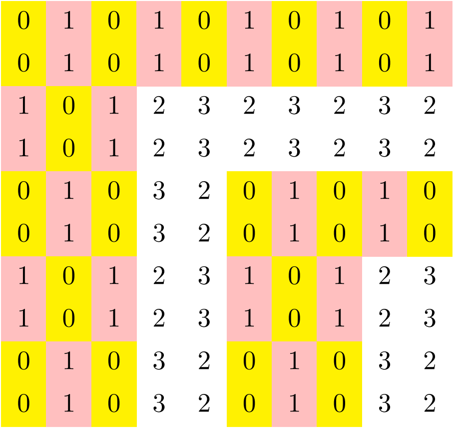
Question 4. *How many of the present methods apply when we add extra transfer options to Lengyel transfer games?*

Mimicing two-move periodicity. We wouldn't expect the vector game $\{(0,-2),(-1,0),(-3,-2),(-2,2)\}$ to have NIM-periodicity, and in fact, it does not. Nor would we expect it to have two-move periodicity, but it sort of does! Observe Figure 15. We see periodicity in the direction $(5,4)$

Question 5. *Why is $(5,4)$ the direction of periodicity of the vector game $\{(0,-2),(-1,0),(-3,-2),(-2,2)\}$? What is the class of vector games that mimic two-move periodicity in this way?*

The most general case. The vector game $\{(0,-3),(-2,0),(-1,3),(-2,2),(-3,1),(-1,-2)\}$ does not exhibit any kind of periodicity that we can pin down, though there are some hints at regularity. For example, the first 9 columns exhibit NIM-periodicity. See Figure 16, where the first failure of NIM-periodicity is highlighted.

Question 6. *Is there any order in the chaos of general subtraction-transfer games?*



0	1	0	1	0	1	0	1	0	1
0	1	0	1	0	1	0	1	0	1
1	0	1	2	3	2	3	2	3	2
1	0	1	2	3	2	3	2	3	2
0	1	0	3	2	0	1	0	1	0
0	1	0	3	2	0	1	0	1	0
1	0	1	2	3	1	0	1	2	3
1	0	1	2	3	1	0	1	2	3
0	1	0	3	2	0	1	0	3	2
0	1	0	3	2	0	1	0	3	2

FIGURE 15. Some of the NIM-values of the vector game $\{(0,-2),(-1,0),(-3,-2),(-2,2)\}$

0	0	2	1	1	3	3	1	1	2	0	0	2	2	1	1	3	0	1	2	0	0	1	2	0	1	2	0	1	1
0	0	2	2	0	3	1	1	3	3	4	0	0	2	2	0	1	1	3	0	4	2	0	1	2	0	0	1	2	0
0	2	2	0	0	2	2	1	3	3	1	1	2	0	0	2	2	1	1	3	0	1	1	2	0	1	2	0	1	2
1	1	3	0	0	2	2	0	0	4	1	1	3	3	0	0	2	2	0	4	1	3	0	1	2	0	1	2	0	0
1	1	3	3	1	2	0	0	2	2	5	5	3	1	1	3	0	0	2	2	5	1	2	0	1	1	2	0	1	2
1	3	3	1	1	3	3	0	2	2	0	0	5	2	1	1	3	0	0	1	2	0	0	1	2	0	1	2	0	1
0	0	2	1	1	3	3	1	1	3	0	0	2	2	4	4	5	1	1	3	0	4	2	0	1	2	0	1	1	2
0	0	2	2	0	3	1	1	3	3	4	1	2	0	0	2	2	4	1	1	3	0	1	2	0	0	1	2	0	1
0	2	2	0	0	2	2	1	3	3	1	1	3	3	0	0	2	2	4	0	1	1	3	0	4	2	0	1	2	0
1	1	3	0	0	2	2	0	0	2	1	1	3	3	1	1	3	0	0	2	2	1	1	3	0	1	2	0	0	1
1	1	3	3	1	2	0	0	2	2	5	0	3	1	1	3	3	5	0	0	2	2	0	1	1	3	0	4	2	0
1	3	3	1	1	3	3	0	2	2	0	0	2	2	1	3	3	1	1	3	0	0	2	2	1	1	3	0	1	1

FIGURE 16. Some of the NIM-values of the vector game $\{(0,-3),(-2,0),(-1,3),(-2,2),(-3,1),(-1,-2)\}$

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