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Generalizing OOOOOOB

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Abstract

We present three versions of the classic two-pile game ONE-OR-ONE-OR-ONE-OF-BOTH generalized to the multi-pile context. In each case, we explore the resulting $\mathcal{P}$-positions. In the first version, there is a simple pattern. In the other two versions, we find partial solutions in each case through two experimental routes. First by limiting the number of piles, then by limiting the number of tokens per pile.

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1. Introduction

The game ONE-OR-ONE-OR-ONE-OF-BOTH, abbreviated OOOOOOB, is a classic puzzle:

> There are two piles of tokens. On their turn, a player may take a token from one pile, take a token from the other pile, or take one token from each pile. The last player to move wins.

Students generally discover the following.

Proposition 1.1. *In* OOOOOOB*, the $\mathcal{P}$-positions are precisely those where both piles are even.*

In this paper, we explore the $\mathcal{P}$-positions of three natural ways to generalize OOOOOOB to more than two piles.

- In Version A, a player selects any non-empty set of piles and removes one token from each.
- In Version B, a player may remove any one token *or* remove one token from each pile.
- In Version C, a player may remove any one token *or* select two piles and remove one token from each.

Version A is easy to solve, and the proof is left as an exercise for the reader.

Proposition 1.2. *In Version A, the $\mathcal{P}$-positions are those where every pile has an even number of tokens.*

Curiously, we can show that the all-even positions are $\mathcal{P}$-positions for every version, without having a full classification of $\mathcal{P}$-positions.

Lemma 1.1. *If all piles are even, then it is a $\mathcal{P}$-position in both Version B and Version C.*

Proof. We proceed by induction on the total number of tokens in the game, and prove both cases simultaneously.

First, note that the unique terminal position is a $\mathcal{P}$-position. Next, let $k > 0$, and consider a position G with all even piles and k total tokens. Let H be an option of G. In Version B, H has either one odd pile, or all odd piles. In Version C, H has one or two odd piles. In each case, there is an option J of H which has all even piles, so by induction, J is a $\mathcal{P}$-position. Thus H is an $\mathcal{N}$-position. Since the choice of H was arbitrary, G is a $\mathcal{P}$-position. □

Corollary 1.1. *If exactly one of the piles is odd, then it is an $\mathcal{N}$-position in both Version B and Version C. In Version B, any position with all odd piles is an $\mathcal{N}$-position. In Version C, any position exactly two odd piles is an $\mathcal{N}$-position.*

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In the next two sections, we investigate the $\mathcal{P}$-positions for Versions B and C. In each case, we obtain partial results in two directions. First, we obtain results for 3, 4, or 5 total piles. Then we see what we can find if we instead limit the maximum pile size to 1, 2, or 3 tokens.

This paper was inspired by recent work of Takayuki Morisawa [5] on m-pile divisor NIM. As in traditional NIM, the winning strategy for the multiple-pile version of that game is a generalization of the winning strategy for the two-pile version. So we wondered, "What about OOOOOOB?"

OOOOOOB can be found, for example, in [3, Chapter 7, Problem 23] or [1, Chapter 3, Section 2.3]. Note that two-pile OOOOOOB is the two-dimensional vector subtraction game defined by $\{(-1, 0), (0, -1), (-1, -1)\}$. See [4] and [2] for more on vector games generally.

We used a computer to generate data for these results. See the last author's website (`http://www.thotsaporn.com`) for the relevant Maple code.

2. Version B

For Version B, we start with two recursive results. Let G, H, J, U, A, etc, denote elements of $\mathbb{N}^k$ for some $k \geq 0$. We use *unioption* and *alloption* to denote options which are obtained by removing a single token and removing one token from each pile, respectively. We use $\mathcal{P}$*-option* (similarly, $\mathcal{P}$*-unioption* and $\mathcal{P}$*-alloption*) to denote an option which is itself a $\mathcal{P}$-position.

Lemma 2.1. *If G is nonempty, $1, 1, G$ is of the same outcome class as G.*

Note that this is false if G is empty, since $1, 1$ is an $\mathcal{N}$-position.

Proof. We proceed by induction on the total number of tokens in G. For the base cases that G has 1 or 2 total tokens, note that 1; $1, 1, 1$; $1, 1$; $1, 1, 1, 1$ are all $\mathcal{N}$-positions, and that 2 and $2, 1, 1$ are $\mathcal{P}$-positions.

Suppose G is a nonempty $\mathcal{P}$-position with at least 3 total tokens. Then the options of $1, 1, G$ are

- $1, G$
- H
- $1, 1, H$

where H is an option of G. We will show that each of these is an $\mathcal{N}$-position.

$1, G$ has the $\mathcal{P}$-option G, and H is an option of G, so both of these are $\mathcal{N}$-positions.

Suppose that J is a $\mathcal{P}$-option of H. If J is an unioption, then by induction, $1, 1, J$ is a $\mathcal{P}$-unioption of $1, 1, H$. If J is the alloption of H, then it is also the alloption of $1, 1, H$. In either case, $1, 1, H$ is an $\mathcal{N}$-position.

Now suppose G is a $\mathcal{N}$-position, and that H is a $\mathcal{P}$-option of G. If H is a unioption, then by induction, $1, 1, H$ is a $\mathcal{P}$-unioption of $1, 1, G$. If H is the alloption of G, then it is also the alloption of $1, 1, G$. In either case, $1, 1, G$ is an $\mathcal{N}$-position. □

Lemma 2.2. *If G has a pile of size of at least 2, then $2, 2, G$ is of the same outcome class as G.*

Proof. We proceed by induction on the total number of tokens in G. For the base cases where G has 2 or 3 total tokens, Lemma 1.1 implies that 2 and $2, 2, 2$ are both $\mathcal{P}$-positions. Thus $2, 1$ and $2, 2, 2, 1$ are both $\mathcal{N}$-positions.

Suppose G is a $\mathcal{P}$-position with a pile of size at least 2 and at least 4 total tokens. The options of $2, 2, G$ are

- $1, 2, G$
- $1, 1, A$
- $2, 2, U$

where A is the alloption and U is a unioption of G. We will show that all of these are $\mathcal{N}$-positions.

By Lemma 2.1, $1, 1, G$ is a $\mathcal{P}$-option of $1, 2, G$, and $1, 1, A$ is an $\mathcal{N}$-position.

Let U' be a $\mathcal{P}$-option of U. Corollary 1.1 implies that U' must have a pile of size 2, so we may use induction to say that $2, 2, U'$ is a $\mathcal{P}$-position. Then if U' is a unioption of U, $2, 2, U'$ is a $\mathcal{P}$-unioption of $2, 2, U$. On the other hand, if U' is the alloption of U, then $1, 1, U'$ is an option of $2, 2, U$, and by Lemma 2.1 is a $\mathcal{P}$-position.

Now suppose G is an $\mathcal{N}$-position, and H is a $\mathcal{P}$-option of G. If H is the alloption, then $1, 1, H$ is the alloption of $2, 2, G$, and by Lemma 2.1, it is a $\mathcal{P}$-position. If H is a unioption, then $2, 2, H$ is a unioption of $2, 2, G$, and by induction is a $\mathcal{P}$-position. In either case, $2, 2, G$ is an $\mathcal{N}$-position. □

2.1 Version B: Few piles

Definition 2.1. *Let e_k (or o_k) denote any even (or odd) pile of size at least k.*

Lemma 2.3. *The $\mathcal{P}$-positions for 3 total piles are as follows.*

- e_2, e_2, e_2
- e_2, o_1, o_1

Proof. By Lemma 1.1 and Corollary 1.1, e_2, e_2 and e_2, e_2, e_2 are $\mathcal{P}$-positions, and e_2, o_1; o_1, o_1; o_1, o_1, o_1 and e_2, e_2, o_1 are $\mathcal{N}$-positions. Thus e_2, o_1, o_1 is a $\mathcal{P}$-position. □

Lemma 2.4. *The $\mathcal{P}$-positions for 4 total piles are as follows.*

- $1, 1, e_2, e_2$
- e_2, e_2, e_2, e_2
- e_2, o_3, o_3, o_3

Remark 2.1. *This lemma implies that there is no version of Lemmas 2.1 and 2.2 for $3, 3, G$. For example, e_2, o_3 is an $\mathcal{N}$-position, while $3, 3, e_2, o_3$ is a $\mathcal{P}$-position*

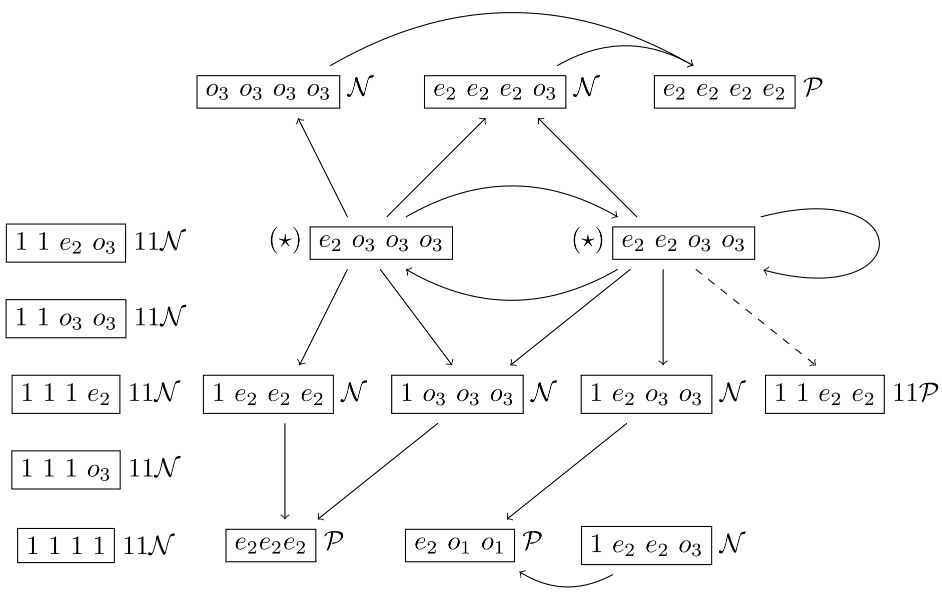

Figure 2.1: All positions of Version B with 4 tokens, along with some relevant options.

Proof. Refer to Figure 2.1. First note that all possible 4-combinations of 1, e_2, o_3 are included. The positions marked $11\mathcal{P}$ and $11\mathcal{N}$ are $\mathcal{P}$- and $\mathcal{N}$-positions, respectively, by Lemma 2.1. The positions marked as $\mathcal{P}$-positions are so by Lemma 1.1 and/or 2.3. The positions marked $\mathcal{N}$ each then have a $\mathcal{P}$-option as indicated.

The remaining positions are the two marked with $(\star)$, e_2, o_3, o_3, o_3 and e_2, e_2, o_3, o_3. We claim that e_2, o_3, o_3, o_3 is always a $\mathcal{P}$-position, and that e_2, e_2, o_3, o_3 is always an $\mathcal{N}$-position. Note that all options of these positions are in the diagram.

We proceed by simultaneous induction on the total number of tokens. The base case is then $2, 2, o_3, o_3$, which has the $\mathcal{P}$-option $1, 1, e_2, e_2$, shown as the dashed arrow in the diagram. Then, in the inductive step, e_2, o_3, o_3, o_3 only has $\mathcal{N}$-options, and e_2, e_2, o_3, o_3 has the $\mathcal{P}$-option e_2, o_3, o_3, o_3. □

Lemma 2.5. *The $\mathcal{P}$-positions for 5 total piles are as follows.*

- $1, 1, e_2, e_2, e_2$
- $1, 1, e_2, o_1, o_1$
- $1, 2, e_2, e_2, o_3$
- $1, e_4, e_4, o_5, o_5$
- e_2, e_2, e_2, e_2, e_2
- e_2, o_3, o_3, o_3, o_3
- e_2, e_2, e_2, o_3, o_3, *with an exception:* $2, e_4, e_4, o_5, o_5$.

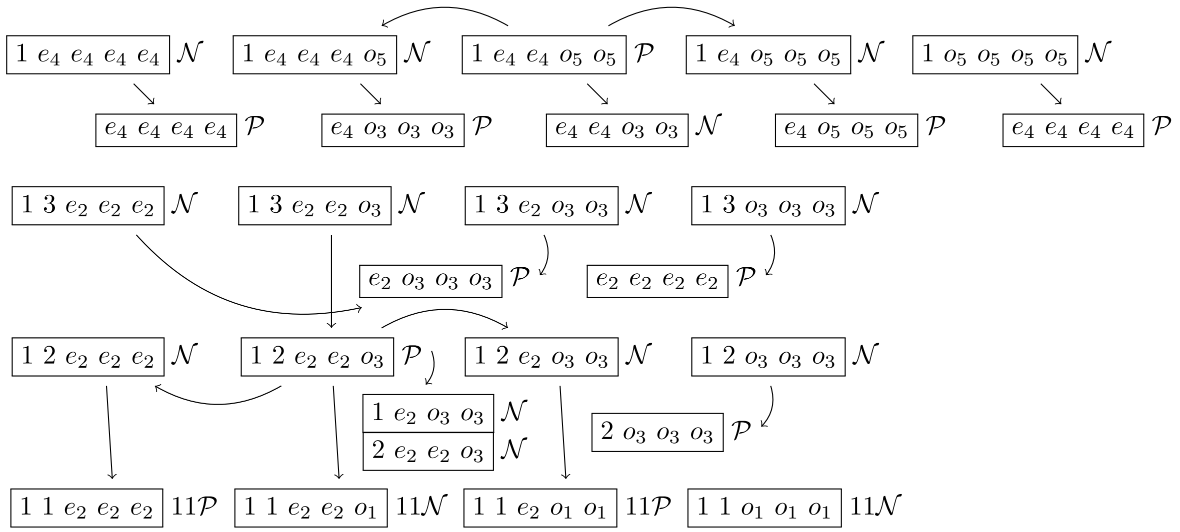

Figure 2.2: All positions of Version B with 5 tokens and at least one pile of size 1, along with some relevant options.

Proof. Our base case is any position with 5 piles where at least one of the piles is 1. Note that these are organized by the size of the second smallest pile in Figure 2.2, in the four rows which start on the far left. The outcome classes of the positions in the bottom row are determined by Lemma 2.1 and Lemma 2.3. After that, for each $\mathcal{N}$-position, a $\mathcal{P}$-option is displayed, and each option of each $\mathcal{P}$-position is displayed. The relevant options which have four piles are justified by the previous lemma.

Assuming now that all piles have size at least 2, we organize the remaining cases as follows. The last four cases subdivide the case with two odds and three evens: one 2 and no other 2s or 3s, one 2 and another 2, one 2 and a 3, no 2s. As a reminder, we mark the claimed outcome class of each. We will analyze the ones marked A first, then the rest together with a simultaneous induction.

- e_2, e_2, e_2, e_2, e_2 – $\mathcal{P}$ – A – by Lemma 1.1
- e_2, e_2, e_2, e_2, o_3 – $\mathcal{N}$ – A – by Corollary 1.1
- e_2, e_2, o_3, o_3, o_3 – $\mathcal{N}$
- e_2, o_3, o_3, o_3, o_3 – $\mathcal{P}$
- o_3, o_3, o_3, o_3, o_3 – $\mathcal{N}$ – A –by Corollary 1.1
- $2, e_4, e_4, o_5, o_5$ – $\mathcal{N}$ – A
- $2, 2, e_2, o_3, o_3$ – $\mathcal{P}$ – A
- $2, 3, e_2, e_2, o_3$ – $\mathcal{P}$
- e_4, e_4, e_4, o_5, o_5 – $\mathcal{P}$

Observe that $2, e_4, e_4, o_5, o_5$ has the $\mathcal{P}$-option $1, e_4, e_4, o_5, o_5$, so it is an $\mathcal{N}$-position.

The only options of $2, 2, e_2, o_3, o_3$ are the previously established $\mathcal{N}$-positions $1, 1, e_2, e_2, o_1$; $1, 2, e_2, o_3, o_3$; $2, 2, e_2, e_2, o_1$ (Corollary 1.1); or the position $2, 2, o_1, o_3, o_3$, which is given by Lemmas 2.2 and 2.3.

We claim that the four remaining cases are $\mathcal{P}$-positions except e_2, e_2, o_3, o_3, o_3, proceeding by induction on the total number of tokens. The base case is then $2, 2, 2, 3, 3$, which is a $\mathcal{P}$-position by Lemmas 2.2 and 2.3.

Next if $e_2, e_2, o_3, o_3, o_3 = 2, 2, o_3, o_3, o_3$, then it has the $\mathcal{P}$-option $1, 1, e_2, e_2, e_2$. Otherwise it has the option $2, o_3, o_3, o_3, o_3$, which, by induction, is a $\mathcal{P}$-position.

We show the remaining cases are $\mathcal{P}$-positions by listing their options (with the alloption last) and noting that they are all $\mathcal{N}$-options. For e_2, o_3, o_3, o_3, o_3, we have o_1, o_3, o_3, o_3, o_3; e_2, e_2, o_3, o_3, o_3; and o_1, e_2, e_2, e_2, e_2. For $2, 3, e_2, e_2, o_3$, we have $1, 3, e_2, e_2, o_3$; $2, 2, e_2, e_2, o_3$; $2, 3, e_2, e_2, e_2$; $1, 2, 3, e_2, o_3$; e_2, e_2, o_3, o_3, o_3; and $1, 2, o_1, o_1, e_2$. For e_4, e_4, e_4, o_5, o_5, we have e_4, e_4, e_4, e_4, o_5 and e_2, e_2, o_3, o_3, o_3. □

The situation for 6 piles is more tedious to verify. Once enough base cases are verified, the general case becomes clear. Hence we leave the proof to the reader.

Lemma 2.6. *The $\mathcal{P}$-positions for 6 total piles are as follows.*

- $1, 1, 1, 1, e_2, e_2$
- $1, 1, e_2, o_3, o_3, o_3$
- $1, 1, e_2, e_2, e_2, e_2$
- $1, 2, e_4, e_4, e_4, o_3$

- $1, 3, e_6, e_6, o_5, o_5$
- $2, 3, e_6, e_6, e_6, o_5$
- $2, e_2, e_2, o_3, o_3, o_3$, *except* $2, 3, e_6, e_6, o_5, o_5$
- $e_2, e_2, e_2, e_2, e_2, e_2$
- $e_4, e_4, e_4, e_4, o_3, o_3$
- $e_2, o_3, o_3, o_3, o_3, o_3$

proof sketch. Suppose sufficiently many base cases have been verified, perhaps by computer. We proceed by induction on the total number of tokens. Suppose we have a position with no ones, twos, or threes, and that the result has been verified for all positions with fewer total tokens. We already know that positions with no odd piles are $\mathcal{P}$-positions. If there are 1 or 6 odd piles, there is an option with no odd piles, so it is an $\mathcal{N}$-position. If there are 3 or 4 odd piles, there is an option with 2 odd piles, so it is an $\mathcal{N}$-position. If there are 2 or 5 odd piles, then there are only options with 1, 3, 4, or 6 odd piles, so it is a $\mathcal{P}$-position. □

Based on our findings for 3, 4, 5, and 6 piles, we conjecture the following. We have verified this by computer up to 11 piles, up to 10 tokens per pile.

Conjecture 2.1. *Suppose we have $k \geq 3$ piles of tokens. Let $m = \lceil \frac{k}{3} \rceil$. Suppose each pile has more than m tokens. Then it is a $\mathcal{P}$-position precisely when:*

- *If k odd, then the number of odd piles is even*
- *If $k = 0 \bmod 4$, then the number of odd piles is* $0,\ 2,\ 4,\ \ldots \frac{k}{2} - 2,\ \frac{k}{2} + 1\ \ldots,\ k-5,\ k-3,\ k-1.$
- *If $k = 2 \bmod 4$, then the number of odd piles is* $0,\ 2,\ 4,\ \ldots \frac{k}{2} - 1,\ \frac{k}{2} + 2\ \ldots,\ k-5,\ k-3,\ k-1.$

2.2 Version B: Few tokens per pile

Lemma 2.7. *Suppose that all piles have at most 3 tokens. In particular, there are a_1 piles of size 1, a_2 piles of size 2, and a_3 piles of size 3. Then the nonempty $\mathcal{P}$ positions are* $(a_1, a_2, a_3) =$

- $(e_0, \geq 1, 0)$
- $(o_1, o_1, 1)$
- $(e_0, o_1, \geq 2)$

Proof. First assume $a_3 = 0$. If $a_2 = 0$ and $a_1 > 0$, then Corollary 1.1 says we have an $\mathcal{N}$-position. So fix $a_2 > 0$ and proceed by induction on the total number of tokens. If $a_1 = 0$, Lemma 1.1 shows that we have a $\mathcal{P}$-position. If $a_1 > 0$ is odd, then by induction, the option to remove a pile of size 1 is a $\mathcal{P}$-position. If $a_1 > 0$ is even, then by induction, all unioptions are $\mathcal{N}$-positions, while Corollary 1.1 shows that the alloption is, as well.

By this first case, if $a_3 > 0$, the alloption is a $\mathcal{P}$-position precisely when a_2 is even, so in the remaining cases, if a_2 even, we have an $\mathcal{N}$-position.

Next assume $a_3 = 1$ and that a_2 is odd. If a_1 is even, taking one token from the pile of size 3 is a $\mathcal{P}$-option. If a_1 is odd, then the possible unioptions are $(a_1, a_2, a_3) = (o_1, e_2, 0), (e_0, e_1, 1), (e_0, o_1, 1)$, each of which is a $\mathcal{P}$-position.

Finally assume $a_3 \geq 2$ and a_2 is odd. If a_1 is odd, then by induction, the option to take one of the piles of size 1 is a $\mathcal{P}$-position. If a_1 is even, then the possible unioptions are $(a_1, a_2, a_3) = (e_0, e_2, 1), (e_0, e_2, \geq 2), (o_1, e_2, \geq 2), (o_1, o_1, \geq 2)$, each of which is a $\mathcal{P}$-position. □

We offer the next two cases without proof. The cases were discovered by computer, and the verification, while long, is routine, and would violate the page limit of this journal.

Lemma 2.8. *Suppose that all piles have at most 4 tokens, and that there is a pile of size 4. In particular, there are a_i piles of size i. Then the nonempty $\mathcal{P}$ positions are as follows.* $(a_1, a_2, a_3, a_4) =$

- *Base cases:* $(o, e, 1, 1)$, $(e, e, \geq 2, 1)$, $(e, e, 0, \{1, 2\})$, $(o, \geq 1, e_4, 2)$, $(e, o, o_3, 2)$, $(e, o, 2, 2)$, $(e, o, 0, \{1, 2, 3\})$, $(o, o, 1, \{2, 3\})$
- *General Cases:* $(e, e, e, \geq 3)$, $(e, o, e, \geq 4)$, $(o, o, o, \geq 4)$

Lemma 2.9. *Suppose that all piles have at most 5 tokens, and that there is a pile of size 5. In particular, there are a_i piles of size i. Then the nonempty $\mathcal{P}$ positions are as follows.* $(a_1, a_2, a_3, a_4, a_5) =$

- *23 base cases:* $(e, e, e, e_4, 2)$, $(e, e, e, o, 2)$,$(e, e, e, 1, \{1, 3\}) - (e, e, 0, 1, 1)$, $(e, e, o, e_4, \{1, 3\})$,$(e, e, o, 1, \{2, 4\})$, $(e, e, o, o, \{1, 3\}) - (e, e, o, 3, 3)$, $(e, o, e, e_4, 4)$, $(e, o, e, 0, \{2, 4\})$, $(e, o, e, \{0, 2\}, o) - (e, o, 0, \{0, 2\}, 1)$, $(e, o, e, o_5, \{2, 4\})$, $(e, o, e, 3, 3)$, $(e, o, o, \{0, 2\}, \{2, 4\})$, $(e, o, o, e_6, 3)$, $(e, o, o, e_4, 1)$, $(e, o, o, 0, \{1, 3\})$, $(e, o, 1, 2, 1)$, $(e, o, o, o_5, \{1, 3\})$, $(o, e, e, 2, 2)$, $(o, e, e, o_5, 3)$, $(o, e, 0, 1, 1)$, $(o, e, o, e_4, 2)$, $(o, e, o, 2, \{1, 3\}) - (o, e, 1, 2, 1)$, $(o, e, o, o_3, 4)$, $(o, o, e, 2, \{2, 4\}) - (o, o, 0, 2, 2)$, $(o, o, 0, 0, 2, 1)$, $(o, o, e, e_4, \{1, 3, 5\}) - (o, o, e, 4, 3)$, $(o, o, e, o_5, \{1, 3\})$, $(o, o, 0, 3, 1)$, $(o, o, o, e_4, 4)$, $(o, o, o, 2, \{1, 3\}) - (o, o, 1, 2, 1)$, $(o, o, o, o_5, 2)$, $(o, o, o, 3, 3)$
- *General Cases:* $(e, e, e, o, \geq 4)$, $(e, e, o, o, \geq 5)$, $(e, o, o, e, \geq 5)$, $(e, o, e, e, \geq 6)$

For piles of size up to 6, we have preliminary calculations which show that the general case for $\mathcal{P}$-positions when $a_6 \geq 8$ is

- $a_5 + a_3$ is even and a_1 is even; or
- $a_5 + a_3$, a_2, and a_1 are all odd

All of the results of this section follow a pattern, and we conjecture accordingly.

Conjecture 2.2. *Let $n \geq 3$. Suppose we have a position with a_i piles of size i for $1 \leq i \leq n$, and that $a_n \geq 2n - 4$. Then the outcome class is a function of the parities of $a_1, a_2, \ldots a_{n-1}$.*

3. Version C

3.1 Version C: Few piles

Definition 3.1. *Suppose $x_i \in \{e, o\}$ for all $1 \leq i \leq n$. Let $\langle x_1, x_2, \ldots, x_n\rangle$ denote the set of all positions $a_1, a_2, \ldots, a_n$ where $a_1 \leq a_2 \leq \ldots \leq a_n$ and x_i is the parity of a_i for all $1 \leq i \leq n$.*

Lemma 3.1. *If there are 3, 4, or 5 piles, the $\mathcal{P}$-positions are as follows, where the piles are listed in nondecreasing order.*

- $\langle e, e, e\rangle$
- $\langle o, o, o\rangle$
- $\langle e, e, e, e\rangle$
- $\langle e, o, o, o\rangle$
- $\langle e, e, e, e, e\rangle$
- $\langle e, e, o, o, o\rangle$
- $\langle o, o, e, e, o\rangle$
- $\langle o, o, o, o, e\rangle$

Proof. For each of 3, 4, and 5 piles, recall that Lemma 1.1 and Corollary 1.1 say that each all-evens position is a $\mathcal{P}$-position, and each position with exactly one or two odd piles is an $\mathcal{N}$-position.

In the case of 3 piles, the only remaining position, $\langle e, e, e\rangle$, is then an $\mathcal{N}$-position.

Next assume there are 4 piles. We prove the remaining cases by induction on the number of tokens. As a base case, note that $\langle 1, 1, 1, 1\rangle$ is an $\mathcal{N}$-position. If there are no odd piles, then taking one from the smallest pile is an $\mathcal{N}$-option. In the cases of $\langle o, e, o, o\rangle$, $\langle o, o, e, o\rangle$, $\langle o, o, o, e\rangle$, taking one from the even pile and one from the smallest odd pile is an $\mathcal{N}$-option. Finally, no options of $\langle e, o, o, o\rangle$ are $\mathcal{N}$-positions.

Next assume there are 5 piles. We proceed by induction, adding the $\mathcal{P}$-position $\langle 0, 0, 1, 1, 1\rangle$ to our collection of base cases. Next, the following are all the remaining claimed $\mathcal{N}$-positions. In each case, a winning move is indicated by taking a token from the pile(s) which are capitalized.

$$\langle e, O, o, o, o\rangle, \langle O, e, o, o, o\rangle, \langle o, o, e, O, o\rangle, \langle o, o, o, E, O\rangle, \langle e, O, E, o, o\rangle, \langle e, O, o, E, o\rangle, \langle E, o, o, o, e\rangle$$

$$\langle o, E, e, O, o\rangle, \langle O, e, o, E, o\rangle, \langle O, e, o, o, E\rangle, \langle o, o, E, o, e\rangle, \langle o, o, o, E, e\rangle, \langle O, O, o, o, o\rangle$$

It is then routine to check that there is no move to from one of the claimed $\mathcal{P}$-positions to another one. □

Unlike the cases for 5 or fewer piles, the case of 6 piles does not follow a strict parity pattern. The data in this case is more chaotic, and as a demonstration, we report the $\mathcal{P}$-positions in the case that there are at least two piles of size 1. Again, we omit the tedious and routine proof.

Lemma 3.2. *Suppose there are six piles, and at least two of the piles has size 1. Then the $\mathcal{P}$-positions are as follows.*

- $\langle 1,1,1,1,1,1\rangle$
- $\langle 1,1,1,1,e,o\rangle$
- $\langle 1,1,1,x,y,y\rangle$, *where x and y have same parity*
- $\langle 1,1,e,e,o,o\rangle$, *last two not equal*
- $\langle 1,1,x,x,y,y+1\rangle$ *with x odd, y even*
- $\langle 1,1,e,o,e,e\rangle$ *with last two not equal*
- $\langle 1,1,x,y,z,z\rangle$ *with x odd, $x \neq y$, y and z same parity*

3.2 Version C: Few tokens per pile

Lemma 3.3. *Suppose that all piles have at most 3 tokens. In particular, there are a_1 piles of size 1, a_2 piles of size 2, and a_3 piles of size 3. Then the $\mathcal{P}$ positions are as follows:*

- $a_1 = 0$, $a_3 = 0$ ($\mathcal{P}$)
- $(a_1, a_2, a_3) = (0,1,3), (0,2,3), (1,0,2), (1,0,5), (2,0,1)$ ($\mathcal{P}$)
- $a_1 + 2 \cdot a_2 = 0 \bmod 3$, *with the following exceptions:*
  - $\star$ $(a_1, a_2, a_3) = (0, 3k, e)$, *where* $e = 1, 2$
  - $\star$ $(a_1, a_2, a_3) = (1, 3k+1, e)$, *where* $e = 0, 1$
  - $\star$ $(a_1, a_2, a_3) = (2, 3k+2, e)$, *where* $e = 0$
  - $\dagger$ $(a_1, a_2, a_3) = (0,0,4), (0,0,5), (1,1,2), (1,1,3), (1,1,4), (1,1,5), (2,2,3), (3,0,1), (3,0,2), (3,0,5)$

0	0	1	2	3	4	5	6
0	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$
1	$\mathcal{P}$	$\star$			$\mathcal{P}$		
2	$\mathcal{P}$		$\star$			$\mathcal{P}$	
3	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$
4	$\mathcal{P}$	$\star$			$\mathcal{P}$		
5	$\mathcal{P}$		$\star$			$\mathcal{P}$	
6	$\mathcal{P}$			$\mathcal{P}$	●		$\mathcal{P}$

1	0	1	2	3	4	5	6
0	$\star$		$\mathcal{P}$	$\dagger$			$\mathcal{P}$
1		$\star$			$\mathcal{P}$		
2			$\mathcal{P}$			$\mathcal{P}$	
3	$\star$			$\mathcal{P}$			$\mathcal{P}$
4		$\star$			$\mathcal{P}$	●	
5			$\mathcal{P}$	●	●	$\mathcal{P}$	
6	$\star$			$\mathcal{P}$			$\mathcal{P}$

2	0	1	2	3	4	5	6
0	$\star$	$\mathcal{P}$		$\dagger$			$\mathcal{P}$
1		$\dagger$			$\mathcal{P}$		
2			$\mathcal{P}$			$\mathcal{P}$	●
3	$\star$			$\mathcal{P}$	●	●	$\mathcal{P}$
4		$\mathcal{P}$	●	●	**P**		
5			$\mathcal{P}$			$\mathcal{P}$	
6	$\star$			$\mathcal{P}$			$\mathcal{P}$

3	0	1	2	3	4	5	6
0	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$
1	$\mathcal{P}$	$\dagger$			$\mathcal{P}$		
2	$\mathcal{P}$		$\dagger$			$\mathcal{P}$	$\mathcal{P}$
3	$\mathcal{P}$			$\mathcal{P}$			
4		$\mathcal{P}$			$\mathcal{P}$		
5			$\mathcal{P}$			$\mathcal{P}$	
6	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$

4	0	1	2	3	4	5	6
0	$\dagger$			$\mathcal{P}$			$\mathcal{P}$
1		$\dagger$			$\mathcal{P}$		
2			$\mathcal{P}$			$\mathcal{P}$	
3	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$
4		$\mathcal{P}$			$\mathcal{P}$		
5			$\mathcal{P}$			$\mathcal{P}$	
6	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$

5	0	1	2	3	4	5	6
0	$\dagger$	$\mathcal{P}$		$\dagger$			$\mathcal{P}$
1		$\dagger$			$\mathcal{P}$		
2			$\mathcal{P}$			$\mathcal{P}$	
3	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$
4		$\mathcal{P}$			$\mathcal{P}$		
5			$\mathcal{P}$			$\mathcal{P}$	
6	$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$

Figure 3.1: Positions of Version C with small piles. In each grid, the number in the top-left corner is a_3, the row is a_2, and the column is a_1. Exceptions to the general case are highlighted. The position $(4,4,2)$ is marked with **P** and its options are marked with ●.

Proof. Figure 3.1 shows all $\mathcal{P}$-positions for $a_1, a_2 \leq 6$, $a_3 \leq 5$. As a visual guide, the options of $(4,4,2)$ are marked with ●. It is then easily checked that the general case holds also for $a_1, a_2 \leq 6$ and $a_3 = 6, 7$. These are enough base cases for the following induction.

By Lemma 1.1 $a_1, a_3 = 0$ gives a $\mathcal{P}$-position. Thus any position with $1 \leq a_1 + a_3 \leq 2$ has a $\mathcal{P}$-option which is all 2s. That is, the columns containing $\star$s only contain $\mathcal{N}$-positions.

Suppose $a_1 + a_3 \geq 3$, and any of a_1, a_2, or a_3 are greater than 5. If $a_1 + 2 \cdot a_2 \neq 0 \bmod 3$, then we may remove either 1 or 2 tokens from piles of size 1 or 2 to obtain $a_1 + 2 \cdot a_2 = 0 \bmod 3$, which, by induction, is a $\mathcal{P}$-option. □

Based on Lemma 3.3, we think that, if there are sufficient piles of each size, Version C mimics the one-dimensional game Subtract(1, 2). We have verified the following for $a_i \leq 5$ and $n \leq 5$.

Conjecture 3.1. *Suppose we have a position of Version C where there are a_1, a_2, $\ldots a_n$ piles of size 1, 2, $\ldots$, n, respectively. Suppose further that $a_i \geq 2$ for all $1 \leq i \leq n$. Suppose further that the position is not precisely 1, 1, 2, 2, 3, 3, 3 (i.e., (2, 2, 3) in the above lemma). Then it is a $\mathcal{P}$-position if and only if the total number of tokens is a multiple of 3.*

Our last lemma shows a kind of counterexample to this conjecture if we allow large gaps in the sequence $a_1, \ldots, a_n$, in particular, if $a_2 = a_3 = \ldots = a_{n-1} = 0$.

Lemma 3.4. *Suppose we have a position with n piles of size 1 and a single pile of size n. Then it is a $\mathcal{P}$-position precisely when*

- $k \geq 2n$ *and* $k + n = 0 \bmod 3$, *or*
- $k < 2n$ *and* $k + 2n = 0 \bmod 4$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	$\boxed{\mathcal{P}}$			$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$
1			$\boxed{\mathcal{P}}$			$\mathcal{P}$			$\mathcal{P}$			$\mathcal{P}$	
2	$\mathcal{P}$				$\boxed{\mathcal{P}}$			$\mathcal{P}$			$\mathcal{P}$		
3			$\mathcal{P}$				$\boxed{\mathcal{P}}$			$\mathcal{P}$			$\mathcal{P}$
4	$\mathcal{P}$				$\mathcal{P}$				$\boxed{\mathcal{P}}$			$\mathcal{P}$	
5			$\mathcal{P}$				$\mathcal{P}$				$\boxed{\mathcal{P}}$		
6	$\mathcal{P}$				$\mathcal{P}$				$\mathcal{P}$				$\boxed{\mathcal{P}}$

Figure 3.2: Positions of Version C with k piles of size 1 and a single pile of size n. The case $k = 2n$ is $\boxed{\text{boxed}}$.

Note that $k = 2n$ implies that both $k + n = 0 \bmod 3$ and $k + 2n = 0 \bmod 4$.

Proof. We refer the reader to Figure 3.2, and leave the details as an exercise. □

From any position on Figure 3.2, the legal moves are up 1, left 1, left 2, up-and-left 1. In other words, it is isomorphic to the two-dimensional vector subtraction game defined by $\{(0, -1), (-1, 0), (-2, 0), (-1, -1)\}$.

4. Applications to impartial game sums

Suppose we have finitely many impartial game positions. If we consider their usual disjunctive sum, where one is allowed to select one of the games and play in it, we obtain the usual theory of $\mathcal{N}$ and $\mathcal{P}$ positions via the NIM-sum.

On the other hand, suppose we have two game positions, G and H, and we are allowed to make a move, either in G, in H, or in both. In this case, we have a $\mathcal{P}$-position if and only if G and H are both $\mathcal{P}$-positions. In other words, this is isomorphic to playing OOOOOOB if we replace $\mathcal{P}$ with 'even' and $\mathcal{N}$ with 'odd.' If we extend this to an arbitrary set of game positions, then this isomorphism still holds for Version A of GENERALIZED OOOOOOB.

Question 4.1. *Is there some game sum operation which gives a similar analogy for Version B or Version C?*

Not for usual sums. For example, $\langle o, o, o \rangle$ is a $\mathcal{P}$-position for Version C, but if we 'play Version C' on the NIM-position 3, 1, 1, we have the $\mathcal{P}$-option 1, 1, 1. Maybe for another kind of sum?

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