Sequences: Polynomial, C-finite, Holonomic, ...

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Jan 22, 2020

Introduction

Mathematics is an art of finding patterns. As a combinatorialist, we deal with a lot of sequences. The process of determining some patterns of given sequences is essentially important. The common ways to explain the sequences are by their recurrence relations or generating functions. Polynomial as sequences, C-finite sequences (constant coefficient linear recurrence) and Holonomic sequences (polynomial coefficient linear recurrence) are the most common ansatzs as pattern searching of each sequence. Here we are on the expedition for a new ansatz that can explain a bigger class of patterns.

Polynomial as Sequences

The sequence $\{a(n)\}_{n=0}^{\infty}$ satisfies some polynomial equation:

$$a(n) = c_k n^k + c_{k-1} n^{k-1} + ... + c_1 n + c_0,$$

where $k, c_0, c_1, ..., c_k$ are fixed numbers.

Example 1: Let $a(n) = 1 + 2 + \cdots + n$. Then $a(n) = \frac{1}{2}n^2 + \frac{1}{2}n$.

For $a(n)$ as a general polynomial of degree k, we notice that the degree of $b(n) := a(n+1) - a(n)$ goes down by 1. Apply this process $k+1$ times, we will obtain the zero sequence,

$$(N-1)^{k+1} a(n) = 0, \quad n \geq 0,$$

where the shift operator $N \cdot a(n) = a(n+1)$.

Polynomial as Sequences

Example 2: $a(n) := n^2 - 2n$ satisfies

$$a(n+3) - 3a(n+2) + 3a(n+1) - a(n) = 0, \quad n \geq 0,$$

where $a(0) = 0, a(1) = -1, a(2) = 0$.

It follows that **the generating function**

$$\sum_{n=0}^{\infty} a(n)x^n = \frac{q(x)}{(1-x)^{k+1}},$$

where $q(x)$ is some polynomial of degree at most k.

Polynomial as Sequences

Some **closure properties** of polynomial as sequences:
Let $a(n)$ and $b(n)$ be a polynomial as sequence of degree r and s.

1. $\{a(n) + b(n)\}_{n=0}^{\infty}$ is also a polynomial as sequence of degree at most $\max(r, s)$.
2. $\{a(n) \cdot b(n)\}_{n=0}^{\infty}$ is also a polynomial as sequence of degree at most $r + s$.
3. $\{\sum_{j=0}^{n} a(j)\}_{n=0}^{\infty}$ is also a polynomial as sequence of degree at most $r + 1$.

C-finite Sequences

The linear recurrence relation with only constant coefficients ([3, 4]) i.e. the sequence $\{a(n)\}_{n=0}^{\infty}$ where there are constants $k, c_1, \ldots, c_{k-2}, c_{k-1}, c_k$ such that

$$a(n) + c_1 a(n-1) + \cdots + c_{k-1} a(n-k+1) + c_k a(n-k) = 0, \quad \text{for all } n \geq k.$$

In this case, we say $a(n)$ satisfies a linear recurrence of order k. The polynomial as sequence is the special case of C-finite sequences.

The generating function

$$\sum_{n=0}^{\infty} a(n) x^n = \frac{q(x)}{1 + c_1 x + c_2 x^2 + \cdots + c_k x^k},$$

where $q(x)$ is some polynomial of degree at most $k-1$.

C-finite Sequences

Examples:

1. (Fibonacci sequence) $a(n) = a(n-1) + a(n-2)$, where $a(0) = 0, a(1) = 1$.
2. The sequence $\{a(n)\}_{n=0}^{\infty}$

$$2, 3, 5, 9, 17, 33, 65, 129, 257, 513, 1025, \ldots$$

satisfies the recurrence relation

$$a(n) = 3a(n-1) - 2a(n-2), \quad n \geq 2.$$

and its generating function can be written as

$$f(x) = \sum_{n=0}^{\infty} a(n)x^n = \frac{-(3x-2)}{(2x-1)(x-1)} = \frac{-(3x-2)}{2x^2-3x+1}.$$

C-finite Sequences

Some **closure properties** of C-finite sequences:
Let $a(n)$ and $b(n)$ be C-finite sequences of order r and s.

1. $\{a(n) + b(n)\}_{n=0}^{\infty}$ is a C-finite sequence of order at most $r + s$,
2. $\{a(n) \cdot b(n)\}_{n=0}^{\infty}$ is a C-finite sequence of order at most rs,
3. $\{\sum_{j=1}^{n} a(j)\}_{n=0}^{\infty}$ is a C-finite sequence of order at most $r + 1$,
4. $\{a(mn)\}_{n=0}^{\infty}, \quad m \in \mathbb{Z}^{+}$ is a C-finite of order at most r.

C-finite Sequences

This closures properties are very useful for proving identities. For example to prove that

$$F_{2n} = 2F_nF_{n+1} - F_n^2, \quad n \geq 0,$$

we only define

$$a(n) := F_{2n} - 2F_nF_{n+1} + F_n^2, \quad n \geq 0$$

and calculate the upper bound for order of $a(n)$. Then we verify numerically that $a(i) = 0, \;\; 0 \leq i \leq d$ where d is the upper bound of order.

Holonomic Sequences

The linear recurrence relation with polynomial coefficients ([3]) i.e. the sequence $\{a(n)\}_{n=0}^{\infty}$ where there are integer k and polynomials $p_0(n), p_1(n), \ldots, p_{k-1}(n), p_k(n)$ ($p_0(n) \neq 0$), each of which has degree at most d such that

$$p_0(n)a(n) + p_1(n)a(n-1) + \cdots + p_{k-1}(n)a(n-k+1) + p_k(n)a(n-k) = 0,$$

for all $n \geq k$.
The C-finite sequence is the special case of holonomic sequence.

Holonomic Sequences

Examples:

1. (Factorial) $a(n) = n \cdot a(n-1)$, where $a(0) = 1$.
2. (Harmonic numbers) $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ satisfies the relation:

$$(n+1)H_n - (2n+3)H_{n+1} + (n+2)H_{n+2} = 0, \quad n \geq 1.$$

The combinatorics people have applied this in their works for quite some time. But it has become very popular since the invent of computer algebra like Macsyma (early 70s) or Maple (early 80s). (Although I wish Maple had this command built-in.) It is the most commonly used ansatz right now.

Holonomic Sequences

A holonomic differential equation (aka holonomic function, D-finite function)
Let $f(x) := \sum_{n=0}^{\infty} a(n)x^n$. Then $f(x)$ satisfies the relation

$$q_0(x)f(x) + q_1(x)f'(x) + \cdots + q_k(x)f^{(k)}(x) = 0,$$

where $q_i(x)$ are some polynomials of degree at most d.

Holonomic Sequences

Some **closure properties** of holonomic sequences:
Let $a(n)$ and $b(n)$ be holonomic sequences and $A(x)$ and $B(x)$ be their generating functions.

1. $\{a(n)+b(n)\}_{n=0}^{\infty}$ is holonomic,
2. (Cauchy product) $A(x)B(x)$ and (Hadamard product) $\{a(n)\cdot b(n)\}_{n=0}^{\infty}$ are holonomic,
3. $\{\sum_{j=1}^{n} a(j)\}_{n=0}^{\infty}$ is holonomic,
4. $\{a(\lfloor un+v \rfloor)\}_{n=0}^{\infty}$ is holonomic for any non-negative rational number u and v.

Polynomial-Recursive Sequences

The sequence $\{a(n)\}_{n=0}^{\infty}$ where there are a polynomial with $r+1$ variables such that

$$P(a(n), a(n-1), \ldots, a(n-r)) = 0 \text{ for all } n \geq r.$$

Example: (Somos sequence) Somos-4

$$a(n) \cdot a(n-4) - a(n-1) \cdot a(n-3) - a(n-2)^2 = 0, \quad n \geq 5$$

where $a(1) = a(2) = a(3) = a(4) = 1$.

Michael Somos is well known for his sequences. To some surprise, the sequence contains only integer values. Several mathematicians have studied the problem of proving and explaining this integer property of the Somos sequences.

X-Recursive Sequences

We introduce the new ansatz that generalize the holonomic ansatz. We first give an example.

X-Recursive Sequences

A hint to Somos-like sequence

Consider a sequence $a(n)$ generated from a nonlinear relation

$$0 = a(n)a(n+1)a(n+3) - a(n)a(n+2)^2 - a(n+2)a(n+1)^2, \quad \text{for all } n \geq 0.$$

where $a(0) = 1, a(1) = 1$ and $a(2) = 2$.
Some of the first terms of this sequence are

$$1, 1, 2, 6, 30, 240, 3120, 65520, 2227680, 122522400, \ldots.$$

This sequence is growing too fast to be C-finite or holonomic, but still simple enough for human to detect the pattern. Can you guess?

X-Recursive Sequences

Answer:

$$a(n) = F_{n+1} \cdot a(n-1), \qquad a_0 = 1,$$

where F_n is the Fibonacci sequence. This example suggests the new type of ansatz that might prove the integrality of Somos-like sequences.

X-Recursive Sequences

X-Recursive Sequence

The sequence $\{a(n)\}_{n=0}^{\infty}$ where there are a linear recurrence with the C-finite sequences coefficients.

$$C_{0,n}a(n) + C_{1,n}a(n-1) + C_{2,n}a(n-2) + \cdots + C_{k,n}a(n-k) = 0$$

where each of the sequences $C_{i,n}, \;\; 0 \leq i \leq k$ are C-finite.

X-Recursive Sequences

More Examples

1. $a(n) = a(n-1) + 2^n a(n-2), \qquad a_0 = a_1 = 1.$
2. $a(n) = F_n \cdot a(n-1) + F_{n-1} \cdot a(n-2), \qquad a(0) = a(1) = 1.$ This sequence is A089126 in Sloane.
3. (Summation) $a(n) = \sum_{i=1}^{n-1} F_i \cdot a(i), \qquad a_1 = 1.$

   Some of the first terms are

   $$1, 1, 2, 6, 24, 144, 1296, 18144, 399168, 13970880.$$

   It is not too hard to show that this sequence fits into the new ansatz.

   $$a(n) = C_n \cdot a(n-1).$$

   where $C_n = 2C_{n-1} - C_{n-3}$ with $C_2 = 3, C_3 = 3$ and $C_4 = 4$.

X-Recursive Sequences

Generating Functions

Example 1: Let $a(n) = F_n \cdot a(n-1)$. Then

$$
\begin{aligned}
f(x) = \sum_{n=0}^{\infty} a(n)x^n = \sum_{n=0}^{\infty} F_n \cdot a(n-1)x^n = x \sum_{n=0}^{\infty} (c_1\alpha_+^n + c_2\alpha_-^n) \cdot a(n-1)x^{n-} \\
= c_1' x \sum_{n=0}^{\infty} \alpha_+^n \cdot a(n)x^n + c_2' x \sum_{n=0}^{\infty} \alpha_-^n \cdot a(n)x^n \\
= c_1' x f(\alpha_+ x) + c_2' x f(\alpha_- x),
\end{aligned}
$$

where α_+ and α_- are the roots of equation $x^2 - x - 1 = 0$.

X-Recursive Sequences

Example 2: Let $a(n) = (n+1)2^n \cdot a(n-1)$. Then

$$f(x) = \sum_{n=0}^{\infty} a(n)x^n = \sum_{n=0}^{\infty}(n+1)2^n \cdot a(n-1)x^n = 2x\sum_{n=0}^{\infty}(n+2)2^n \cdot a(n)x^n$$

$$= 2x^2\sum_{n=0}^{\infty} n2^n \cdot a(n)x^{n-1} + 4x\sum_{n=0}^{\infty} 2^n \cdot a(n)x^n$$

$$= 2x^2 f'(2x) + 4xf(2x).$$

X-Recursive Sequences

Closure Properties of X-recursive sequences Let $a(n)$ and $b(n)$ be X-recursive sequences.

1. $\{a(n)+b(n)\}_{n=0}^{\infty}$ is X-recursive,
2. (Cauchy product) $c(n) := \sum_{k=0}^{n} a(k)b(n-k)$ and (Hadamard product) $\{a(n)\cdot b(n)\}_{n=0}^{\infty}$ are X-recursive,
3. $\{\sum_{j=1}^{n} a(j)\}_{n=0}^{\infty}$ is X-recursive,
4. $\{a(\lfloor un+v \rfloor)\}_{n=0}^{\infty}$ is X-recursive for any non-negative rational number u and v.

How to Guess

The biggest problem of this ansatz is the calculation. We demonstrate the calculation by using the example where $a(n)$ satisfies linear recurrence of order 2

$$a(n) = C_{1,n}a(n-1) + C_{2,n}a(n-2), \quad n \geq 2,$$

and that $C_{1,n}$ and $C_{2,n}$ are C-finite of order 2. We then need solve the system of equations

$$\begin{aligned} a(n) &= C_{1,n}a(n-1) + C_{2,n}a(n-2), \quad 2 \leq n \leq N, \\ C_{1,n} &= c_1 C_{1,n-1} + c_2 C_{1,n-2}, \quad 4 \leq n \leq N, \\ C_{2,n} &= d_1 C_{2,n-1} + d_2 C_{2,n-2}, \quad 4 \leq n \leq N. \end{aligned}$$

Total we have $(N-2) + (N-4) + (N-4) = 3N - 10$ equations and $2(N-2) + 4 = 2N$ variables. Therefore we must choose $N > 10$ in order to guess the patterns.

How to Guess

The reader may already notice that these are the system of non-linear equations. It seems like the computation is taking too long for any practical purpose.

Some Remarks

At the end, we did not go very far with this new ansatz. The first problem is the slowness causing by solving the system of non-linear equations. This makes the guessing very difficult. The second problem is that the other well known non-holonomic sequences like Bell numbers, Somos-4,5,6 do not seem to fall in this class. So it is not as helpful as we might want it to be. Nonetheless it is useful to keep this in mind, it could be helpful when we need it one day.

DD-finite Generating Functions

During my presentation on X-recursive sequences in 2017, I was informed by Christoph, my colleague from RISC, Austria that there were a group of people at RISC leading by Veronika Pillwein working on similar idea, [2]. But instead of generalizing the sequence, she generalized the generating function.

DD-finite Generating Functions

DD-finite functions
Let $f(x) := \sum_{n=0}^{\infty} a(n)x^n$. $f(x)$ is called a DD-finite function if $f(x)$ satisfies the relation

$$q_0(x)f(x) + q_1(x)f'(x) + \cdots + q_k(x)f^{(k)}(x) = 0,$$

for some constant k and D-finite functions $q_i(x)$, $0 \leq i \leq k$.

DD-finite Generating Functions

Some **examples** of these functions are

1. $e^x f(x) - f'(x) = 0$. Here $f(x) = C \cdot e^{e^x}$. The coefficient satisfies the relation

$$(n+1)a(n+1) = \sum_{k=0}^{n} \frac{a(k)}{(n-k)!}.$$

We note that if $C = e^{-1}$, $f(x)$ is the (exponential) generating function of the famous Bell numbers, $b(n)$.

By substituting $a(n) =: b(n)/n!$, we have the relation

$$b(n+1) = \sum_{k=0}^{n} \binom{n}{k} b(k), \quad b(0) = 1.$$

DD-finite Generating Functions

2. $f(x) - \sin(x)\cos(x)f'(x) = 0$. This time $f(x) = C\tan(x)$. From this equation, the coefficients $a(n)$ of $f(x)$ satisfies the relation

$$(1-n)a(n) = \sum_{k=1}^{n-1} kc_{n-k+1}a_k, \quad a_0 = 0, a_1 = 1,$$

where $c_k = \dfrac{-4}{k(k-1)}c_{k-2}, \quad c_0 = 0, c_1 = 1.$

3. $((x-1)e^x + 1)f(x) + x(e^x - 1)f'(x) = 0$. Here $f(x) = \dfrac{x}{e^x - 1}$ which is the exponential generating function of the Bernoulli numbers. From this equation, the coefficients $a(n)$ of $f(x)$ satisfies the relation

$$a(n) = -\sum_{k=0}^{n-1} \frac{a(k)}{(n+1-k)!}.$$

DD-finite Generating Functions

These three classic sequences satisfy the differential equations of order 1.

The **closure properties** of DD-finite functions was derived in [2] too. Let f, g be DD-finite functions of orders r and s. Then

1. $f + g$ is a DD-finite with order at most $r + s$.
2. fg is a DD-finite with order at most rs.

DD-finite Generating Functions

It is suggested, from the examples, that the ansatz for the sequences should be

$$\sum_{j=0}^{n} c_{n,j,0}\, n(n-1)\ldots(n-k+1)a(n) + \ldots + \sum_{j=0}^{n} c_{n,j,k}\, a(n-k) = 0.$$

where, for each l, $c_{n,j,l}$ are 2-dimensional holonomic sequences in n and j. However giving a sequence, it is still difficult to determine whether its generating functions is DD-finite. We runs into the same problem for solving system of non-linear equations as in the X-recursive case.

References

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