HITTING k PRIMES BY DICE ROLLS

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MUIC Math Seminar

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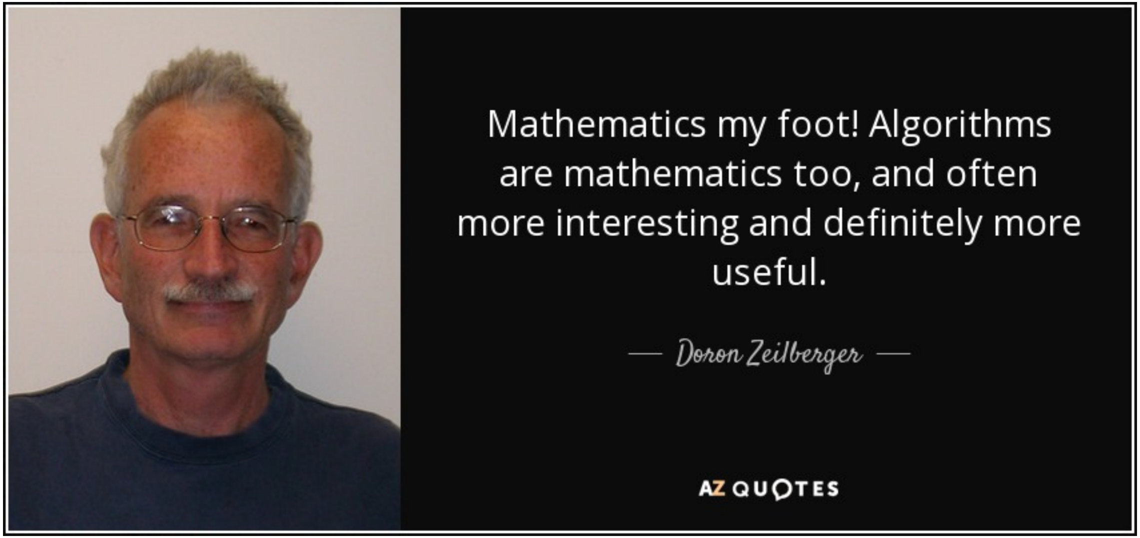
Doron Zeilberger turnning 75

This year my Ph.D. advisor turns 75. I want to use this opportunity to mention about him and talk about his recent work.



Dr.Z (as students call him) is a very kind person. I was very lucky to study with him during my Ph.D. years. He also has a good vision about how mathematics going to be. Some of the things he said, at the time, create some controversial. But looking back at what he said he is usually right.

This picture is him and his servant (sometimes co-author) Shalosh B. Ekhad.

Some of his famous opinions

- (1995) People who believe that Applied Math is Bad Math are Bad Mathematicians
- (1997) Show Up To The Weekly Colloquium and Try to Behave!
- (1999) Don't Ask: What Can The Computer do for ME?, But Rather: What CAN I do for the COMPUTER?
- (2006) Why P Does Not Equal NP and Why Humans Will Never Prove It by Themselves

- (2008) Twenty Pieces of Advice for a Young (and also not so young) Mathematician
  - Learn to use and write programs in Maple (or Mathematica, or any computer algebra system)
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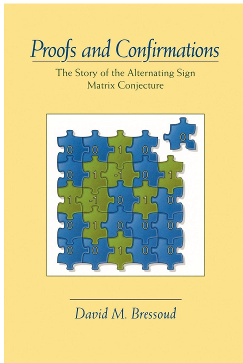
  - ...
- (2011) Mathematics is Indeed a Religion, But It has too Many Sects! Let's Unite Under the New God of Experimental Mathematics

His selected works

- Zeilberger's Algorithm, the method of creative telescoping. This is an automated algorithm to evaluate the binomial sum. For example

$$\sum_k \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2.$$

- Proof of the alternating sign matrix conjecture, which David Bressoud wrote the book about.



- A Proof Of George Andrews' and David Robbins' q-TSPP Conjecture (that won him *David P. Robbins Prize* in 2016 along with Koutschan and Kauers)

- Recently he is the world record holder for irrationality measure of π through the paper

  The Irrationality Measure of Pi is at most 7.103205334137...

  (written with Zudilin).

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Part I

Warm up problem:

Question 1: On average, how many times you need to roll a fair die until the sum of the numbers that you roll is an odd number?

Question 2: You would like to roll a fair die until the sum of the numbers that you roll is an odd number. What is the average of the odd number that you stopped with.

How many rolls to get a prime?

The first part of this talk will follow the talk by *Yaakov Malinovsky*, University of Maryland, Baltimore County, USA joint work with *Noga Alon*, Princeton University, USA and Tel Aviv University, Israel.

Question 1: On average, how many times you need to roll a fair die until the sum of the number you rolls is *a prime number*?

Question 2: Also, let τ be the number of times you rolls until the sum is a prime number. What do you think the distribution of τ would look like?

Let's try to get some ideas about the answers through simulation.

Computing Mean and Variance

We first note about the common identities to calculate the mean and the second moment that

$$E[\tau] = \sum_{k \geq 1} Pr(\tau \geq k), \qquad \text{and} \qquad E[\tau^2] = \sum_{k \geq 1} (2k-1) Pr(\tau \geq k).$$

For (rigorous) values of $E[\tau]$ and $Var(\tau)$, we will concentrate on computing the partial sum of the above equations.

Let X_i be the value from each dice roll. That is $X_i \in \{1, 2, 3, 4, 5, 6\}$. Also we let $S_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1$.

For each integer $k \geq 1$ and integer n, we define

$$\boxed{p(k, n) = Pr\left(X_1 + X_2 + \cdots + X_k = n \quad \text{and} \quad X_1 + X_2 + \cdots + X_i \notin P \text{ for all } i \leq k\right),}$$

where $P := \{2, 3, 5, 7, 11, \ldots\}$, i.e. P is the set of all primes.

Step 1: Dynamic-Programming Algorithm

We now apply the dynamic programming to compute $p(k, n)$.

With suitable condition and initial condition:

- $p(k, n) = 0$ if $n < 0$ or n is prime, and
- $p(0, n) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise.} \end{cases}$

We compute $p(k, n)$ recursively as

$$\boxed{p(k, n) = \frac{1}{6} \sum_{i=1}^{6} p(k-1, n-i).}$$

Step 2: Collect all of possible values of n

We have,

$$\boxed{Pr\left(\tau \geq k\right) := p(k) = \sum_{n=k-1}^{6(k-1)} p(k-1, n).}$$

Lower bound of $E[\tau]$ and $Var(\tau)$

We see that by fixing a big K,say 1000, we find a good approximation of $E[\tau]$ and $Var(\tau)$.

That is

$$E[\tau] \approx \sum_{k=1}^{K} Pr(\tau \geq k) = \sum_{k=1}^{K} p(k).$$

Similarly,

$$E[\tau^2] \approx \sum_{k=1}^{K} (2k-1) Pr(\tau \geq k) = \sum_{k=1}^{K} (2k-1) p(k).$$

By computation on Maple, we have

$$E_{200} := \sum_{k=1}^{200} p(k) = 2.428497913693504228528923 1\ldots,$$

and

$$Var_{200} := \sum_{k=1}^{K} (2k-1)Pr(\tau \geq k) - E_{200}^2 = 6.2427786682790745679619273\ldots.$$

Bounds for the remainders

We defined the remainders of the mean and second moment as

$$R_K = E[\tau] - E_K \quad \text{and} \quad R_K^{(2)} = E[\tau^2] - E_K^{(2)}.$$

Numerically, R_k and $R_K^{(2)}$ seem to be very small. Noga Alon and Yaakov Malinovsky beautifully formally bounded these values by applying the *prime number Theorem.*

They showed that

$$R_{1000} < 7 \cdot 10^{-8} \quad \text{and} \quad R_{1000}^{(2)} < \frac{1}{10,000}.$$

Please see his slides from page 12 on.

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Part II

The second part of this talk we will follow the paper, *How Many Dice Rolls Would It Take to Hit Your Favorite Kind of Number?,* by Lucy MARTINEZ and Doron ZEILBERGER.

This paper is inspired by the work in the first part. For this talk, the *Favorite Kind of Number* is a prime number as in the first part.

The paper applies **symbolic computation** through the idea of *multi-variate generating function.*

Using Symbolic Computation to Model the Game

Sooner or later (with probability 1) the game ends, at some number of rounds, when you reached a certain prime.

Let $q(k, n)$ be the probability that it ended after k rounds and that the running sum then was the prime n.

Everything about this process is encoded in the *bivariate probability generating function*, the infinite double-series

$$\boxed{F(t, x) := \sum_{k=1}^{\infty} \left(\sum_{n=k}^{6k} q(k, n) x^n \right) t^k,}$$

n are primes.

Of course $F(1,1) = 1$, and the expected duration is $F_t(1,1)$, while the expected final location is $F_x(1,1)$. The variance, higher moments, and mixed moments could be gotten by differentiating with respect to t and/or x and then substituting $x = 1, t = 1$.

Alas, this is an infinite series, so let's be more modest and try and compute the **truncated series,** for a given finite maximal number of rounds, R:

$$\boxed{F_R(t,x) := \sum_{k=1}^{R} \left(\sum_{n=k}^{6k} q(k,n) x^n \right) t^k .}$$

Illustrate how to get Maple to compute $F_R(t,x)$

Let $P(x)$ be the probability generating function of the die:

$$P(x) = \frac{1}{6}\sum_{i=1}^{6} x^i.$$

We also define the *prime operator* on polynomials $\sum_{i=1}^{n} a_i x^i$.

$$\mathcal{P}\left(\sum_{i=1}^{n} a_i x^i\right) = \sum_{\substack{1\le i\le n \\ i \text{ prime}}} a_i x^i.$$

For example,

$$\mathcal{P}(x + 3x^2 + 5x^4 + \frac{1}{2}x^6 + 8x^7) = 3x^2 + 8x^7.$$

To compute $F_R(t, x)$, we introduce the auxiliary sequence of polynomials $S_R(x)$, $D_R(x)$, that takes care of the *survivors* and *death (hit a prime)* at the R^{th} round.

Initialize: $S_0(x) := 1$. Also $F_0(t, x) := 0$.

Suppose that you already have $F_{R-1}(t, x)$. If currently you are at the R^{th} round, with the previous survival polynomial, $S_{R-1}(x)$, define

$$\begin{aligned} D_R(x) &:= \mathcal{P}(P(x)S_{R-1}(x)), \\ S_R(x) &:= P(x)S_{R-1}(x) - D_R(x), \\ F_R(t, x) &:= F_{R-1}(t, x) + D_R(x)t^R. \end{aligned}$$

Conditional probability generating function

The probability that the game ends in $\leq R$ rounds is $F_R(1,1)$, (for R large this is very close to 1). Also of interest is the conditional probability generating function

$$\bar{F}_R(t,x) := \frac{F_R(t,x)}{F_R(1,1)}.$$

Implementation

We will show these computations in Maple.

- Probability of getting primes in ≤ 200 rolls $= F_{200}(1, 1)$
  $= 0.9999999999999999997097984795591060572739485923327 5 \ldots.$
- Expectation of number of rolls, conditional on ≤ 200 rolls $= \dfrac{\partial \bar{F}_{200}(t, x)}{\partial t}\Big|_{\substack{t=1,\\ x=1}}$
  $= 2.428497913693504171193372778103882509502277781699 1 \ldots.$
- Expectation of exit locations, conditional on ≤ 200 rolls $= \dfrac{\partial \bar{F}_{100}(t, x)}{\partial x}\Big|_{\substack{t=1,\\ x=1}}$
  $= 8.499742697927264592371464814856285448333190376324 2 \ldots.$

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Part III

In this last part, we will follow another paper of Dr.Z which the title is the title of this talk:

HITTING k PRIMES BY DICE ROLLS

by NOGA ALON, YAAKOV MALINOVSKY, LUCY MARTINEZ, AND DORON ZEILBERGER

Here we consider τ_r which is the random variable whose value is the number of dice rolls required until the accumulated sum equals primes r times.

In part I, we showed that the expected value of τ_1 is close to 2.4284979136935.

Calculation

To make the notation consistence with part I, we let k be number of rolls and r be the number of times to hit primes.

The calculation goes as following:

Let $p(k, n, r)$ be the probability of rolling the die k times such that the sum $X_1 + X_2 + ... + X_k = n$ and all partial sum is prime less than r times.

We apply the dynamic programming to compute $p(k, n, r)$.

Dynamic programming

With suitable condition and initial condition:

- Let $r' = \begin{cases} r-1, & \text{if } n \text{ is prime} \\ r, & \text{otherwise.} \end{cases}$
- $p(k,n,r) = 0$ if $n < 0$ or $r' \leq 0$, and
- $p(0,n,r) = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{otherwise.} \end{cases}$

We then compute $p(k,n,r)$ recursively as

$$\boxed{p(k,n,r) = \frac{1}{6}\sum_{i=1}^{6} p(k-1, n-i, r').}$$

Similar to earlier, we let τ_r be the random variable of number of rolls to first hit primes r times

$$\boxed{Pr\left(\tau_r \geq k\right) := p(k,r) = \sum_{n=k-1}^{6(k-1)} p(k-1,n,r).}$$

Lastly,

$$E[\tau_r] \approx \sum_{k=1}^{K} Pr(\tau_r \geq k) = \sum_{k=1}^{K} p(k,r).$$

Similarly,

$$E[\tau_r^2] \approx \sum_{k=1}^{K} (2k-1) Pr(\tau_r \geq k) = \sum_{k=1}^{K} (2k-1) p(k,r).$$

Some computational results for fixed r

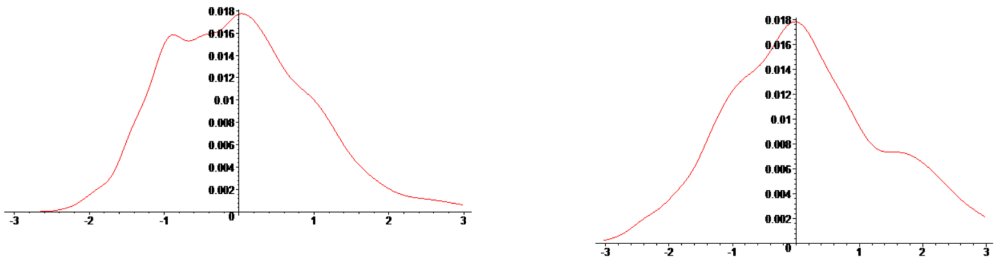
Let's see some statistics when $r = 10$ and 20 (with $K = 200$). For the full tables, please consult with the paper, [3].

$$\begin{aligned} E[\tau_{10}] &= 42.76471867007900, & E[\tau_{20}] &= 100.2614629822323 \\ std(\tau_{10}) &= 13.98233569093723, & std(\tau_{20}) &= 22.66095353683931 \\ Skew(\tau_{10}) &= 0.7974491605443765, & Skew(\tau_{20}) &= 0.5261802925527392 \\ Ku(\tau_{10}) &= 3.998920889672180, & Ku(\tau_{20}) &= 3.376825850225309. \end{aligned}$$

Here $Skew = \dfrac{E[(\tau_r - \mu_r)^3]}{std(\tau_r)^3}$ and Ku is short for Kurtosis defined by $Ku = \dfrac{E[(\tau_r - \mu_r)^4]}{std(\tau_r)^4}$.

Distributions for fixed r

We end this part with some figures of the **scaled** probability density functions for the number of rolls of a fair die until visiting the primes r times for $r = 20, 25$. (Recall that the scaled version of a random variable X with expectation μ and variance σ^2 is $\frac{(X-\mu)}{\sigma}$).



The distribution of τ_r for $r = 20$ and 25.

Asymptotic results

Apart of the wonderful results that we presented, another main result of the paper is that, for large r, the expected value of τ_r is $(1+o(1))r\log_e r$, where the $o(1)$-term tends to zero as r tends to infinity.

I will not go through the whole proofs. Instead, I will highlight a few parts of the proof that I like.

Remark from the paper: If we substitute for $\pi(n)$ its approximation $\dfrac{n}{\log n}$ and repeat the analysis, $E[\tau_r]$ that follows is $r(\log r + \log\log r + O(1))$.

The heuristic expression $f(r) = r(\log r + \log\log r + 0.543) + 8.953$. For the record, here are the ratios of $E[\tau_r]/f(r)$ for $r = 20, 40, 60, 80, 100$, respectively:

$$0.9861651120, 0.9976101939, 0.9966486957, 0.998338113, 0.9997448512.$$

Lemma 2.1

In the plain language, Lemma 2.1 stated that

If we keep rolling a die and let S be the set of partial sum of die rolls i.e $X_1 + X_2 + \cdots + X_i \in S, \quad i \geq 1$. For any positive integer x, let $p(x)$ denote the probability that S hits x. Then $p(x)$ converges to the constant $\dfrac{2}{7}$ with an exponential rate.

Proposition 2.3

Another key ingredient of the proof is to apply **Chernoff's bound** to the tail of the distribution.

Here "x_i hit" means the partial sum of rolling a die equals to x_i. Proposition 2.3 stated that

For any sequence $x_1 < x_2 < \cdots < x_n$ of positive integers and any $a \geq \sqrt{n}\log(n)$,

$$Pr\left(\left|\#x_i \text{ hit } - \frac{2}{7}n\right| \geq a\right) \leq e^{-c'\frac{a^2}{n\log(n)}},$$

for some absolute positive constant c'.

References

[1] Noga Alon and Yaakov Malinovsky, *Hitting a prime in 2.43 dice rolls (on average)* .

[2] Lucy Martinez, and Doron Zeilberger, *How many Dice Rolls Would It Take to Reach Your Favorite Kind of Number?*, Maple Transactions, vol.3, No. 3 (Autumn 2023).

[3] Noga Alon, Yaakov Malinovsky, Lucy Martinez, and Doron Zeilberger, *Hitting k primes by dice rolls.*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