# Statistics of Domino Tilings on a Rectangular Board

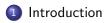
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Thotsaporn 'Aek' Thanatipanonda (Science [Statistics of Domino Tilings on a Rectangular

July 3, 2018 1 / 31



Fibonacci numbers and Statistics on an 2-by-*n* board



# Introduction

It is well known the Fibonacci sequence,  $F_n$ , is the number of ways to cover a 2-by-(n-1) board using only the horizontal(H) or vertical(V) 2-by-1 dominos.

It is natural to generalize this idea to a rectangular m-by-n board where m is a fixed number and n is symbolic.

We can try harder and consider the mixed moment  $E[V^aH^b]$  for fixed non-negative integers a, b but general m, n. After all these moments will give an information of the distribution of "V-H statistics".

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### Pibonacci numbers and Statistics on an 2-by-n board



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# Fibonacci numbers and Statistics on an 2-by-n board

We fixed the board to size 2-by-n.

Let  $V_n$  be a random variable of the number of vertical dominos on the tiling of the board of size 2-by-*n*.

First consider the power sum,  $S[V^r] := \sum_{b \in B_n} V(b)^r$ . For r = 0, the number of possible tilings on the 2-by-*n* board,  $S[V^0] = f_n$ , where  $f_n$  is the (n + 1)-th Fibonacci numbers.

Next consider the straight moment,  $E[V^r] = S[V^r]/S[V^0] = S[V^r]/f_n.$ 

The average number of V, vertical domino, on a 2-by-n board,  $\mu_n := E[V] = S[V]/f_n$ .

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# Fibonacci numbers and Statistics on an 2-by-n board

Eventually we are interested in the moment about the mean,  $E[(V - \mu)^r]$ and then the scaled-moments  $\frac{E[(V - \mu)^r]}{E[(V - \mu)^2]^{r/2}}$  (which can show the normality of an asymptotic distribution).

The generating function of a random variable V is defined by

$$F_n(v) := \sum_{b \in B_n} v^{V(b)}.$$

For example, on the board of size 2-by-3,  $F_3(v) = v^3 + 2v$ .

 $S[V^0] = F_n(1)$  and  $S[V^r]$  is obtained by applying the operator  $(v \frac{d}{dv})^r$  to  $F_n(v)$ , then substitute v = 1.

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Next, we define the grand generating function H(v, t) by

$$H(v,t)=\sum_{n=0}^{\infty}F_n(v)t^n.$$

In this problem, we have

$$H(v,t) = \frac{1}{1 - vt - t^2},$$
(1)

which can be derived from a simple recurrence

$$H(v, t) = 1 + vtH(v, t) + t^{2}H(v, t).$$

Thotsaporn 'Aek' Thanatipanonda (Science [Statistics of Domino Tilings on a Rectangular

July 3, 2018 8 / 31

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With equation (1) in hand, we calculate data of  $S[V^r]$  (and hence  $E[V^r]$ ) very fast. Recall that

$$\sum_{n=0}^{\infty} S[V^0]t^n = H(1,t) = \frac{1}{1-t-t^2}.$$

In general,

$$\sum_{n=0}^{\infty} S[V^r]t^n = (v\frac{d}{dv})^r H(v,t)|_{v=1}.$$

From the quotient rule in calculus, we find the fact that

$$\sum_{n=0}^{\infty} S[V^r]t^n = (v\frac{d}{dv})^r H(v,t)|_{v=1} = \frac{P_r(t)}{(1-t-t^2)^{r+1}}$$

where  $P_r(t)$  is a polynomial in t of degree at most 2r.

Hence  $S[V^r]$  in fact satisfies the recurrence of the form  $(N^2 - N - 1)^{r+1}$ and can be written in the form,

$$S[V^r] = A(n)f_n + B(n)f_{n-1}$$

where A(n) and B(n) are polynomial in *n* of degree at most *r*.

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We then be able to use computer program to conjecture these formulas by trying to fit the polynomial to the data. Given  $\phi = \frac{1 + \sqrt{5}}{2}$ .

$$f_n := S[V^0] = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

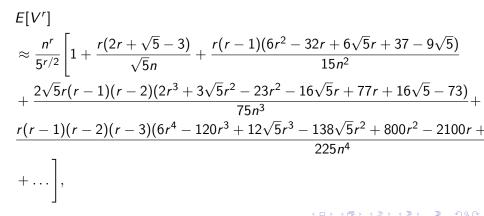
$$\mu_n := E[V] = \frac{1}{f_n} \left[ \frac{n}{5} f_n + \frac{2}{5} (n+1) f_{n-1} \right] \approx \frac{n}{5} + \frac{2}{5\phi} (n+1),$$

$$\begin{split} \mathsf{E}[V^2] &= \frac{1}{f_n} \left[ \frac{n(5n+12)}{25} f_n + \frac{4(n+1)}{25} f_{n-1} \right] \\ &\approx \frac{n(5n+12)}{25} + \frac{4(n+1)}{25\phi}. \end{split}$$

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From these calculations, we are able to make a general conjecture of  $E[V^r]$  starting from the leading terms (and after simplify  $\phi$ ):

#### Conjectures



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We use the conjectures of straight moments to calculate the moment about the mean:

$$\begin{split} E[(V-\mu)^0] &= 1, \\ E[(V-\mu)^1] &= 0, \\ E[(V-\mu)^2] &= E[X^2] - E[X]^2 = \frac{4\sqrt{5}n + 4\sqrt{5} - 8}{25}, \\ E[(V-\mu)^3] &= E[X^3] - 3E[X^2]E[X] + 2E[X]^3 = \frac{8\sqrt{5}n}{125} + \frac{8\sqrt{5}}{125} - \frac{48}{125}, \\ E[(V-\mu)^3] &= \frac{48n^2}{125} + \frac{96n}{125} - \frac{272\sqrt{5}n}{625} + \frac{16}{25} - \frac{272\sqrt{5}}{625}, \\ E[(V-\mu)^5] &\approx \frac{64n^2}{125} + \frac{128n}{125} - \frac{736\sqrt{5}n}{625} + \frac{9776}{3125} - \frac{144\sqrt{5}}{125}, \\ E[(V-\mu)^6] &\approx \frac{192\sqrt{5}n^3}{625} + \dots, \end{split}$$

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These data lead us to the conjectures of general formulas of moment about the mean which we will prove formally.

$$E[(V - \mu)^{2r}] = \left(\frac{2}{5\sqrt{5}}\right)^r \frac{(2r)!n^r}{r!} + \text{ smaller terms},$$
$$E[(V - \mu)^{2r+1}] = \frac{2}{15} \left(\frac{2}{5\sqrt{5}}\right)^r \frac{(2r+1)!n^r}{(r-1)!} + \text{ smaller terms}.$$

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# Toward the Proof

We will prove the conjectures on  $E[V^r]$  and  $E[(V - \mu)^r]$  from the previous slide. Then use them to conclude the asymptotic normality distribution of V. The method of guess and check will again play an important role here.

The generating function  $F_n(v)$  can be defined by the recurrence

$$F_n(v) = vF_{n-1}(v) + F_{n-2}(v) \ , n \ge 2,$$

where

$$F_0(v) = 1, \ F_1(v) = v.$$

# Toward the Proof

The centralized probability generating function of  $F_n(v)$  is

$$G_n(\mathbf{v}) := \sum_i p(i) \mathbf{v}^{i-\mu} = \frac{1}{f_n \mathbf{v}^{\mu}} F_n(\mathbf{v}),$$

The recurrence will look like

$$G_n(v) = v \frac{f_{n-1}G_{n-1}(v)}{f_n v^{\mu_n - \mu_{n-1}}} + \frac{f_{n-2}G_{n-2}(v)}{f_n v^{\mu_n - \mu_{n-2}}} , n \ge 2,$$
(2)

where

$$G_0(v) = 1, \quad G_1(v) = 1.$$

We will use this recurrence to set up some relations for the proof.

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# Probability Background Interim

But before diving into the proof, we need to discuss some necessary probability background.

Definition (Exponential moment generating function)  

$$\phi(t) := E[e^{tX}] = \begin{cases} \sum_{x} e^{tx} p(x) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x) dx & \text{if } x \text{ is continuous.} \end{cases}$$

<u>Note</u>  $m_n := E[X^n] = \phi^n(0).$ 

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# Probability Background Interim

We then have that

$$\phi(t) := E[e^{tX}] = \sum_{x} \sum_{n} \frac{t^{n} x^{n}}{n!} \rho(x) = \sum_{n} \frac{t^{n}}{n!} \sum_{x} x^{n} \rho(x) = \sum_{n} \frac{t^{n}}{n!} m_{n}.$$

Moment of the standard normal distribution:

$$\phi(t) = e^{\frac{t^2}{2}} = \sum_{r} \frac{t^{2r}}{r!2^r}$$
 implies that  $m_{2r} = \frac{(2r)!}{r!2^r}$  and  $m_{2r-1} = 0$ .

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# Main Theorem

#### Back to the main theorem

Theorem

Let 
$$E_r(n) = \frac{E[(V-\mu)^r]}{r!}$$
 then  

$$E_{2r}(n) = \left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^r}{r!} + \text{ smaller terms},$$

$$E_{2r+1}(n) = \frac{2}{15} \left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^r}{(r-1)!} + \text{ smaller terms}$$

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**Proof** We see that

 $G_n(e^t) = \sum_i p(i)e^{t(i-\mu)} = \phi(t)$ , where the random variable  $X = i - \mu$ .

Now define the Maclaurin series of  $G_n(e^t)$  by

$$G_n(e^t) = \sum_r E_r(n)t^r.$$

By the fact of probability generating function mentioned earlier,  $E_r(n) = \frac{E[(V - \mu)^r]}{r!}.$ 

The recurrence (2) becomes:

$$G_n(e^t) = e^t \frac{f_{n-1}G_{n-1}(e^t)}{f_n e^{t(\mu_n - \mu_{n-1})}} + \frac{f_{n-2}G_{n-2}(e^t)}{f_n e^{t(\mu_n - \mu_{n-2})}} , n \ge 2.$$
(3)

We apply an induction on r using (3) to show the assertion of  $E_r(n)$ .

$$G_{n}(e^{t}) = \frac{f_{n-1}}{f_{n}} \left[ \sum_{r=0}^{\infty} \frac{(1-\mu_{n}+\mu_{n-1})^{r} t^{r}}{r!} \right] \cdot \left[ \sum_{r=0}^{\infty} E_{r}(n-1) t^{r} \right] \\ + \frac{f_{n-2}}{f_{n}} \left[ \sum_{r=0}^{\infty} \frac{(-\mu_{n}+\mu_{n-2})^{r} t^{r}}{r!} \right] \cdot \left[ \sum_{r=0}^{\infty} E_{r}(n-2) t^{r} \right].$$

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By comparing coefficient of  $t^r$ , we obtain the relations:

$$\begin{split} E_r(n) &- \frac{f_{n-1}}{f_n} E_r(n-1) - \frac{f_{n-2}}{f_n} E_r(n-2) \\ &= \frac{f_{n-1}}{f_n} a_n E_{r-1}(n-1) + \frac{f_{n-2}}{f_n} b_n E_{r-1}(n-2) \\ &+ \frac{f_{n-1}}{f_n} \frac{a_n^2}{2} E_{r-2}(n-1) + \frac{f_{n-2}}{f_n} \frac{b_n^2}{2} E_{r-2}(n-2) \\ &+ \frac{f_{n-1}}{f_n} \frac{a_n^3}{6} E_{r-3}(n-1) + \frac{f_{n-2}}{f_n} \frac{b_n^3}{6} E_{r-3}(n-2) + \dots \end{split}$$
  
where  $a_n = 1 - \mu_n + \mu_{n-1} \approx 1 - \frac{1}{\sqrt{5}}$  and  $b_n = -\mu_n + \mu_{n-2} \approx -\frac{2}{\sqrt{5}}.$ 

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## Case1: even case: Left hand side:

$$E_{2r}(n) - \frac{f_{n-1}}{f_n} E_{2r}(n-1) - \frac{f_{n-2}}{f_n} E_{2r}(n-2)$$
  
=  $\left(\frac{f_{n-1}}{f_n} + 2\frac{f_{n-2}}{f_n}\right) \left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^{r-1}}{(r-1)!} + \text{ smaller terms}$ 

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#### Right hand side is

$$\begin{aligned} &\frac{f_{n-1}}{f_n} a_n E_{2r-1}(n-1) + \frac{f_{n-2}}{f_n} b_n E_{2r-1}(n-2) + \\ &\frac{f_{n-1}}{f_n} \frac{a_n^2}{2} E_{2r-2}(n-1) + \frac{f_{n-2}}{f_n} \frac{b_n^2}{2} E_{2r-2}(n-2) + \text{ smaller terms} \\ &= \left(\frac{f_{n-1}}{f_n} \frac{a_n^2}{2} + \frac{f_{n-2}}{f_n} \frac{b_n^2}{2}\right) \left(\frac{2}{5\sqrt{5}}\right)^{r-1} \frac{n^{r-1}}{(r-1)!} + \text{ smaller terms.} \end{aligned}$$

which are equal.

**Case2: odd case:** ... (can be done similarly).

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# Asymptotic Distribution

To show the normality of V, we show that  $\frac{V_n - \mu_n}{\sigma_{V_n}} \sim N(0, 1)$  as  $n \to \infty$ . We verify, as  $n \to \infty$ ,  $m_{2r} = \frac{(2r)!}{2^r r!}$  for every r and  $m_{2r-1} = 0$ .

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# Asymptotic Distribution

#### Corollary

The distribution of number of V on a tiling of 2-by-n board is asymptotically normal.

Proof.  $\frac{E[(V_n - \mu_n)^{2r}]}{E[(V_n - \mu_n)^2]^r} = \frac{\left(\frac{2}{5\sqrt{5}}\right)^r \frac{n^r}{r!} (2r)!}{\left(\frac{4n}{5\sqrt{5}}\right)^r} = \frac{(2r)!}{2^r r!},$ and  $\frac{E[(V_n - \mu_n)^{2r+1}]}{E[(V_n - \mu_n)^{2}]^{(r+1/2)}} = 0,$ 

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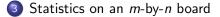
# Asymptotic Distribution

# **Remark** The conjectures of the straight moment $E[V^r]$ can be verified similarly with less calculation.

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Fibonacci numbers and Statistics on an 2-by-*n* board



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Statistics on an *m*-by-*n* board

The method we used for a 2-by-n board can be generalized to an m-by-n board for a fixed m and symbolic n.

# References

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