Practice Problems for Midterm Exam

1. a) Determine whether the statement is true or false.
   Let $\mathbf{v}, \mathbf{u}, \mathbf{w}$ be any vectors in 3-dimensional space.

   i) _____ If $\mathbf{v} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{w}$ and $\mathbf{v} \neq 0$, then $\mathbf{u} = \mathbf{w}$.

   ii) _____ If $\mathbf{u}$ is a unit vector that is parallel to a nonzero vector $\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = \pm\|\mathbf{v}\|$.

   iii) _____ If $\mathbf{v} \times \mathbf{u} = \mathbf{v} \times \mathbf{w}$ and $\mathbf{v} \neq 0$, then $\mathbf{u} = \mathbf{w}$.

   b) Determine whether the points $P_1(6, 9, 7), P_2(9, 2, 0)$ and $P_3(12, -5, -6)$ lies on the same line.

2. a) Find the acute angle of intersection between the two planes

$$2x - 4y + 4z = 6 \qquad \text{and} \qquad 6x + 2y - 3z = 4.$$

b) The planes

$$x + 2y - 2z = 3 \qquad \text{and} \qquad 2x + 4y - 4z = 7$$

are parallel. Find the distance between these two planes.

3. a) Find two unit vectors orthogonal to both $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - 2\mathbf{k}$.

b) Find the area of the parallelogram with vertices $P_1 = (1, 2, 3)$, $P_2 = (1, 3, 6)$, $P_3 = (3, 8, 6)$ and $P_4 = (3, 7, 3)$.

4. a) Where does the line through $(1, 0, 1)$ and $(4, -2, 2)$ intersect the plane $x + y + z = 6$?

b) Find an equation of the plane that contains the line

$$x = 1 + 2t, y = 2 + t, z = -1 + 3t$$

and also contains the origin.

5. a) Show that the curve $\mathbf{r} = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j} + t\mathbf{k}, \quad t \geq 0$, lies on the cone $z = \sqrt{x^2 + y^2}$. Describe the curve.

b) Which of the following is the vector function of the circle of radius 10 centered at $(1, 2)$ which has the arc length function, $s(t)$, satisfies $\dfrac{ds}{dt} = 1$.

i) $\mathbf{r}_1(t) = \langle 1 + 10\cos t, 2 + 10\sin t\rangle$
ii) $\mathbf{r}_2(t) = \langle 1 + 10\cos 10t, 2 + 10\sin 10t\rangle$
iii) $\mathbf{r}_3(t) = \langle 1 + 10\cos\frac{t}{10}, 2 + 10\sin\frac{t}{10}\rangle$
iv) $\mathbf{r}_4(t) = \langle 1 + 10\cos 5t, 2 + 10\sin 5t\rangle$
v) $\mathbf{r}_5(t) = \langle 10\cos t, 10\sin t\rangle$.

6. a) Let $\mathbf{r}_1(t) = \arctan(t)\mathbf{i} + \sin(t)\mathbf{j} + t^2\mathbf{k}$ and $\mathbf{r}_2(t) = (t^2 - t)\mathbf{i} + (2t - 2)\mathbf{j} + (\ln t)\mathbf{k}$.

i) Show that the graph of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ intersect at the origin.

ii) Find the acute angle between the tangent lines to the graphs of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at the origin.

b) Find the length of the curve

$$\mathbf{r}(t) = \langle 1, 3e^t, 4e^t \rangle, 0 \leq t \leq \ln 3.$$

Answers

1. a) F,T,F. b) $P_1P_2 = (6,9,7) + t(3,-7,-7)$, we see that $P_3(12,-5,-6)$ does not lie on this line. The answer is NO.

2. a) We find the angle between normal vectors:
   $\cos\theta = \dfrac{\langle 2,-4,4\rangle \cdot \langle 6,2,-3\rangle}{\|\langle 2,-4,4\rangle\| \, \|\langle 6,2,-3\rangle\|} \rightarrow \theta = \arccos(-4/21)$.
   The acute angle between two planes is $\pi - \arccos(-4/21)$.
   b) Pick any point on either plane and apply the formula i.e. $(3,0,0)$ from the first plane. $d = \dfrac{|-1|}{\sqrt{2^2+4^4+4^2}} = \dfrac{1}{6}$.

3. a) $\pm\dfrac{-2\mathbf{i}+4\mathbf{j}-\mathbf{k}}{\sqrt{21}}$;
   b) $A = |\langle 0,1,3\rangle \times \langle 2,5,0\rangle| = |\langle -15,6,-2\rangle| = \sqrt{265}$.

4. a) $\mathbf{v} = \langle 1,0,1\rangle + \langle -3,2,-1\rangle t$ and intersect at $t=-2$ which is $(x,y,z) = (7,-4,3)$.
   b) $7x - 5y - 3z = 0$.

5. a) $\sqrt{x^2+y^2} = \sqrt{(t\cos(t))^2 + (t\sin(t))^2} = t = z$. The curve forms a spiral shape with expanding radius.
   b) the answer is (iii) because the magnitude of $\mathbf{r}_3'(t)$ is always 1.

6. a) i) $\mathbf{r}_1(0) = \mathbf{r}_2(1) = \langle 0,0,0\rangle$, ii) $\dfrac{\pi}{6}$ b) 10.