The arithmetic-periodicity of CUT for $\mathcal{C} = \{1, 2c\}$

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Impartial, Normal-Play, Games

CUT is a class of partition games played on a finite number of finite piles of tokens. Each version of cut is specified by a cut-set $\mathcal{C}$. A legal move consists of selecting one of the piles and partitioning it into $d+1$ nonempty piles, where $d \in \mathcal{C}$. No tokens are removed from the game.

Two players: Impartial game (both player have the same options) with normal play (the player who makes the last legal move wins).

Example 1

$S = \{1, 4\}$ $\quad n = 7$

Player 1 → [1, 1, 1, 1, 3]

Player 2 → [1, 1, 1, 1, 1, 2]

Player 1 → [1, 1, 1, 1, 1, 1, 1]   Player 1 wins.

Definitions

Definition

Given a pile of size n, an *option* O_n is a sequence of piles obtainable via a single legal move on that pile. In other words, O_n is a particular partition of n with $d+1$ parts for some $d \in \mathcal{C}$. Formally, $O_n = (h_0, h_1, ..., h_d)$ where $1 \leq h_i$ and $h_0 + h_1 + \cdots + h_d = n$. Let $\mathcal{O}_n$ denote the set of all options O_n.

Definition

The nim-value of the option $O_n = (h_0, h_1, ..., h_d)$ is

$$\mathcal{G}(O_n) = \mathcal{G}(h_0) \oplus \mathcal{G}(h_1) \oplus \cdots \oplus \mathcal{G}(h_d).$$

Finally the Sprague-Grundy Theorem tells us that the nim-value $\mathcal{G}(n)$ is given as

$$\mathcal{G}(n) = \text{mex}\{\mathcal{G}(O_n) \mid O_n \in \mathcal{O}_n\},$$

where $\text{mex}(S)$ is the least natural number missing from S.

Example 2

$\mathcal{S} = \{1, 4\}$

Find $g(7)$ Given that

n	1	2	3	4	5	6
$g(n)$	0	1	0	1	2	3

$7 = 6+1,\ 5+2,\ 3+4$

$= 1+1+1+1+3,\ 1+1+1+2+2$

$g([6,1]) = g(6) \oplus g(1) = 3 \oplus 0 = 3$

$g([5,2]) = g(5) \oplus g(2) = 2 \oplus 1 = 3$

$g([3,4]) = g(3) \oplus g(4) = 0 \oplus 1 = 1$

$g([1,1,1,1,3]) = g(1) \oplus \dots \oplus g(1) \oplus g(3) = 0$

$g([1,1,1,2,2]) = 0$

$g(7) = \text{mex}\{3, 3, 1, 0, 0\} = 2$ ✓

Previously known results

Some known results of CUT from Dailly, Duchêne, Larsson, Paris (2020). (CUT was defined pretty recently.)

Cut-set $\mathcal{C}$	Nim sequence
$1 \notin \mathcal{C}$	$(0)^c(+1)$ $\overbrace{0,0,0,0}^{c},\overbrace{1,1,1,1}^{c},\overbrace{2,2,2}^{c}$ where c is the smallest element in $\mathcal{C}$
$1 \in \mathcal{C}$ and $\mathcal{C}$ contains only odd numbers	$(0,1)$ $n \mid 1\ 2\ 3$ $g(n) \mid 0,1,0,1,0,1$
$\{1,2,3\} \subseteq \mathcal{C}$	$(0)(+1)$ (i.e. $\mathcal{G}(n) = n-1$)
$\mathcal{C} = \{1,3,2c\}$	$(0,1)^c(+2)$

All nim sequences here are *arithmetic-periodic* (AP).

Third example   Assume $1 \in G$, G contains only odd number.

To show $g(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$

Case 1 n is odd

$$G_n = \underbrace{[\overbrace{odd, odd, \ldots, odd}^{odd}, \overbrace{even, \ldots, even}^{odd}]}_{\text{even parts}}$$

by induction on n.

$$g(G_n) = \underbrace{0 \oplus \ldots \oplus 0}_{odd} \oplus \underbrace{1 \oplus 1 \oplus \ldots \oplus 1}_{odd} = 1$$

$$g(n) = \text{mex}\{1, 1, 1, \ldots, 1\} = 0.$$

Case 2 n is even ...

Open Problem

$\{1, 2c\}$ $c \geq 2$

Examples from DDLP on page 4

Table 1
Some partition games for which the purely arithmetic-periodicity is proved with the AP-test.

CUT-set	Sprague–Grundy sequence
$\{1, 4\} \cup K$ with $K \subseteq \{6, 8, 10\}$	$((0, 1)^2(2, 3)^2, 1, 4, 5, 4, (3, 2)^2(4, 5)^2(6, 7)^2)\ (+8)$
$\{1, 6\} \cup K$ with $K \subseteq \{8, 10\}$	$((0, 1)^3(2, 3)^3, 1, 4, (5, 4)^2(3, 2)^3(4, 5)^3(6, 7)^3)\ (+8)$
$\{1, 8\}$	$((0, 1)^4(2, 3)^4, 1, 4, (5, 4)^3(3, 2)^4(4, 5)^4(6, 7)^4)\ (+8)$
$\{1, 10\}$	$((0, 1)^5(2, 3)^5, 1, 4, (5, 4)^4(3, 2)^5(4, 5)^5(6, 7)^5)\ (+8)$

Open Problem 1. Given $c \geq 2$, the game $G(\mathcal{C})$ with $\mathcal{C} = \{1, 2c\}$ is arithmetic-periodic of length $12c$ and saltus 8.

$\{1, 4\}$

$\{1, 6\}$

0, 1, 0, 1

0, 1, 0, 1, 0, 1

2, 3, 2, 3

2, 3, 2, 3, 2, 3,

1, 4, 5, 4

1, 4, 5, 4, 5, 4

3, 2, 3, 2

3, 2, 3, 2, 3, 2

4, 5, 4, 5

4, 5, 4, 5, 4, 5

6, 7, 6, 7

6, 7, 6, 7, 6, 7

Our contribution

The periodicity of each CUT $\mathcal{C}$ could be confirmed numerically (i.e. Theorem 9, DDLP and also in WWI, page 90). Hence the periodicity of $\mathcal{C} = \{1, 2c\}, \quad c = 2, 3, 4, 5$ has already been verified. It is left to show for general c:

Theorem ((Main Target) Ellis and T. 2022)

The nim-sequence of the game CUT *with cut-set* $\mathcal{C} = \{1, 2c\}$ *for any* $c \geq 2$ *is precisely*

$$(0,1)^c(2,3)^c, 1, 4, (5,4)^{c-1}, (3,2)^c(4,5)^c(6,7)^c(+8).$$

Outline of the proof: Main Idea 1: Nim-set

The whole proof is quite long. I will highlight the important ideas/observations that we used.

$g(G_n) = g(h_1) \oplus g(h_2) \oplus \dots \oplus g(h_p)$

Nim-set

Definition

The nim-set, $\mathcal{N}(n, p, \mathcal{C})$, is the set of nim values that arise from breaking n tokens into p piles. Formally,

$$\mathcal{N}(n, p, \mathcal{C}) = \{\mathcal{G}(O_n) \mid O_n = \overbrace{(h_1, h_2, h_3, \dots, h_p)}^{p \text{ parts}} \text{ where } h_1 + h_2 + \dots + h_p = n\}.$$

Instead of just looking at the nim values $\mathcal{G}(n)$, we look at the nim-set which these values are calculated from.

Main Idea 1: Nim-set: examples

Note that each nim-value in this set,
$\mathcal{G}(O_n) = \mathcal{G}(h_1) \oplus \mathcal{G}(h_2) \oplus \cdots \oplus \mathcal{G}(h_p)$, is still calculated recursively according the actual rules of the game, that is, using the whole cut-set $\mathcal{C}$.
As an initial observation, note that $\mathcal{N}(n, 1, \mathcal{C}) = \{\mathcal{G}_{\mathcal{C}}(n)\}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\mathcal{N}(n,2,\{1,6\})$	-	{0}	{1}	{0}	{1}	{0}	{1}	{0,2}	{1,3}	{0,2}	{1,3}
$\mathcal{N}(n,7,\{1,6\})$	-	-	-	-	-	-	{0}	{1}	{0}	{1}	{0}
$\mathcal{G}_{\{1,6\}}(n)$	0	1	0	1	0	1	2	3	2	3	2

n	12	13	14	15	16	17	18	19
$\mathcal{N}(n,2,\{1,6\})$	{0,2}	{3}	{0,1,2}	{0,1,3,4}	{0,1,2,5}	{0,1,3,4}	{0,1,2,5}	{0,1,4}
$\mathcal{N}(n,7,\{1,6\})$	{1}	{0,2}	{1,3}	{0,2}	{1,3}	{0,2}	{1,3}	{0,1,2}
$\mathcal{G}_{\{1,6\}}(n)$	3	1	4	5	4	5	4	3

The first 19 nim-sets of CUT for $\mathcal{C} = \{1, 6\}$

Patterns, Patterns, Patterns!!!

After all, if the NIM-sequences for, let's say, $\mathcal{C} = \{1, 8\}$ is going to be a "shift-expanded" version of the NIM-sequence for $\mathcal{C} = \{1, 6\}$, then there is probably a correspondence between their NIM-sets, right?

Yes, but not how you might expect. In fact, it is probably more than you expect...

For example, the initial values of the sequence $\mathcal{N}(n, 3, \{1, 6\})$ are

{}, {},
{0}, {1}, **{0}**, **{1}**, {0}, {1},
{0, 2}, {1, 3}, **{0, 2}**, **{1, 3}**, {0, 2}, {1, 3},
{0, 1, 2}, {0, 1, 3, 4}, **{0, 1, 2, 5}**, **{0, 1, 3, 4}**, {0, 1, 2, 5}, {0, 1, 3, 4},

and the initial values of the sequence $\mathcal{N}(n, 3, \{1, 8\})$ are

{}, {},
{0}, {1}, **{0}**, **{1}**, **{0}**, **{1}**, {0}, {1}
{0, 2}, {1, 3}, **{0, 2}**, **{1, 3}**, **{0, 2}**, **{1, 3}**, {0, 2}, {1, 3},
{0, 1, 2}, {0, 1, 3, 4}, **{0, 1, 2, 5}**, **{0, 1, 3, 4}**, **{0, 1, 2, 5}**, **{0, 1, 3, 4}**, {0, 1, 2, 5}, {0, 1, 3, 4}

But $p = 3$ corresponds to $2 \in \mathcal{C}$, which is in the ruleset of Neither game!

In fact, this correspondence holds for all $c \geq 3$, and all $p \geq 2$!

Main Idea 2: Exiting and entering partitions

When we further analyze the sequence of nim-sets for $p = 2$, we find that it is made up of alternating subsequences.

$g([1,1])$ $g([7,1])$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathcal{N}(n, 2, \{1, 6\})$	-	0	1	0	1	0	1	0	1	0	1	0			
								2	3	2	3	2	3	2	3
														1	0
														0	1
															4

Main Idea 2: Exiting and entering partitions

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\mathcal{N}(n,2,\{1,6\})$															
	2	3	2												
	1	0	1	0											
	0	1	0	1	0	1	0	1	0						
	5	4	5	4	5	4	5	4	5						
					3	2	3	2	3	2	3	2	3	2	3
						6	7	6	7	6	7	6	7	6	7
											0				
											1	0	1	0	1
											4	5	4	5	4
													0	1	0

This is quite striking! There are underlying subsequences which alternate between a and $a \oplus 1$. Furthermore, these subsequences enter and exit the sequence at very particular partitions.

Main Observation 1

Lemma

Assume **main target** *holds up to n. Suppose* $p \geq 4$ *and* $\mathcal{C} = \{1, 2c\}, c \geq 2$. *Then each exiting partition in* $\mathcal{N}(n, p, \mathcal{C})$ *has the same nim-value as some non-exiting partition in* $\mathcal{N}(n, p, \mathcal{C})$.

So for $p \geq 4$, once one of these subsequences starts, it never ends. Formally:

Corollary (Cor 11 in the paper)

Assume **main target** *holds up to n. For* $p \geq 4, c \geq 2$, *if* $v \in \mathcal{N}(n, p, \{1, 2c\})$, *then* $v \oplus 1 \in \mathcal{N}(n + 1, p, \{1, 2c\})$.

Main Observation 1

*

Corollary (Cor 13 in the paper)

Assume **main target** *holds up to n. For $p \geq 4$ and $c \geq 2$,*

$$\mathcal{N}(n+1, p+1, \{\underline{1, 2c}\}) = \mathcal{N}(n, p, \{\underline{1, 2c}\}).$$

Main result and Main Observation 2

$\mathcal{C} = \{1, 2c\}$

The following theorem proved Open problem 1 from DDLP.

Theorem

For $c \geq 3$

$$\mathcal{G}_{\{1,2c\}}(n) = \mathcal{G}_{\{1,6\}}(\phi_1(n)).$$

Note that $\phi_1(2cq + r) = 6q + r'$.

This theorem strongly based on the following definition and lemma.

Main Observation 2

Definition

Given $1 \le r \le 2c$, define $r', 1 \le r' \le 6$ by

$\{1,2c\}$ r	1	2	3	4	5	6	7	...	$2c-3$	$2c-2$	$2c-1$	$2c$
$\{1,6\}$ r'	1	2	3	4	3	4	3	...	3	4	5	6

Next suppose $p \ge 1$, $c \ge 3$, and $n \ge p$. Write $n = k + p - 1 = 2cq + r + p - 1$ with $1 \le r \le 2c$. Then for each $p \ge 1$, $n \ge p$, set

$$\phi_p(n) = \phi_p(2cq + r + p - 1) = 6q + r' + p - 1$$

Main Observation 2: Strong lemma

We first prove that the ϕ correspondence holds on the corresponding nim-sets. This is quite surprising, as it is true for all p, not just $p = 2$ and $p = 2c + 1$.

Lemma

For $c \geq 3$ and $k \geq 1$, $p \geq 2$,

$$\mathcal{N}(k+p-1, p, \{1, 2c\}) = \mathcal{N}(\phi_p(k+p-1), p, \{1, 6\}).$$

This is, in my opinion, the longest part of the paper. We spent about 2 pages verifying the properties of ϕ. Then spending another 1.5 pages for this lemma.

An extension

Lemma (Lem 18 in the paper)

For $p \geq 4, c \geq 2,$

$$\mathcal{N}(n, p+2, \{1, 2c\}) \subset \mathcal{N}(n, p, \{1, 2c\}).$$

Proof.

$$\begin{aligned} \mathcal{N}(n, p+2, \{1, 2c\}) &= \mathcal{N}(n-2, p, \{1, 2c\}) && \text{Corollary 13 twice} \\ &\subset \mathcal{N}(n, p, \{1, 2c\}) && \text{Corollary 11 twice} \end{aligned}$$

□

An extension (continued)

Theorem (Thm 19 in the paper)

Let $\mathcal{C}_1 = \{1, 2c_1, 2c_2, 2c_3, ...\}$, $c_1, c_2, ... \geq 2$ *and* $\mathcal{C}_2 = \{1, 2c\}$ *where* $c = \min\{c_1, c_2, c_3, \ldots\}$. *Then for* $n \geq 1$,

$$\mathcal{G}_{\mathcal{C}_1}(n) = \mathcal{G}_{\mathcal{C}_2}(n).$$

Proof. We proceed by induction on n. For the base case, it is clear that $\mathcal{G}_{\mathcal{C}_1}(1) = \mathcal{G}_{\mathcal{C}_2}(1) = 0$. For the induction step, we assume the statement is true for all $n' < n$. This means that for any $p > 1$, $\mathcal{N}(n, p, \mathcal{C}_1) = \mathcal{N}(n, p, \mathcal{C}_2)$. Hence

$$\begin{aligned}
\mathcal{G}_{\mathcal{C}_1}(n) &= \operatorname{mex}\left\{\mathcal{N}(n, 2, \mathcal{C}_1) \cup \bigcup_{2c_i \in \mathcal{C}_1} \mathcal{N}(n, 2c_i + 1, \mathcal{C}_1)\right\} \\
&= \operatorname{mex}\left\{\mathcal{N}(n, 2, \mathcal{C}_2) \cup \bigcup_{2c_i \in \mathcal{C}_1} \mathcal{N}(n, 2c_i + 1, \mathcal{C}_2)\right\} \\
&= \operatorname{mex}\{\mathcal{N}(n, 2, \mathcal{C}_2) \cup \mathcal{N}(n, 2c + 1, \mathcal{C}_2)\} && \text{Lemma 18} \\
&= \mathcal{G}_{\mathcal{C}_2}(n)
\end{aligned}$$

What's left?

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In this section, we categorize the families of cut sets for which the nim-sequence of CUT remains unknown. There are 4 such families. Let X to be a non-empty set of even numbers, each of which is at least 4. Let Y to be a non-empty set of odd numbers, each of which is at least 5. Let $x = 2c$ and y be the smallest elements of X and Y, respectively.

Family A: $\mathcal{C} = \{1, 3\} \cup X$, or $\mathcal{C} = \{1, 3\} \cup X \cup Y$

We already know from [3, Propositions 8] that the nim-sequence for $\mathcal{C} = \{1, 3, 2c\}$ is $(0, 1)^c(+2)$.

Conjecture 1. *The nim-sequence for all games of* CUT *in this family are all precisely* $(0, 1)^c(+2)$.

Family B: $\mathcal{C} = \{1\} \cup X \cup Y$

The nim-sequence of this family seems to have some resemblance to the nim-sequence for $\mathcal{C} = \{1, 2c\}$ when $c \geq 2$, but we cannot make a full conjecture at this time. The following partial extension of Theorem 19 seems to be true.

DDLP

Conjecture 2. *If* $3x < y$, *then* $\mathcal{G}_{\mathcal{C}}(n) = \mathcal{G}_{\{1,x\}}(n)$ *for* $n \geq 1$.

We note that proving the arithmetic-periodicity of Families A and B would imply Conjecture 1 of [2].

Family C: $\{1, 2\} \subseteq \mathcal{C}, 3 \notin \mathcal{C}, \mathcal{C} \neq \{1, 2\}$.

It is not so clear how to categorize the patterns of this family. However, we do observe:

Conjecture 3. *The nim-sequence for all games of* CUT *in Family C are all* ultimately *arithmetic-periodic.*

Family D: $\mathcal{C} = \{1, 2\}$

The first 36 terms of the nim-sequence for this game of CUT are

0,	1,	2,	3,	1,	4,	3,	2,	4,	5,	6,	7,
8,	9,	7,	6,	9,	8,	11,	10,	12,	13,	10,	11,
13,	12,	15,	14,	16,	17,	5,	15,	17,	16,	19,	18

It is supposed that the nim-sequence for this particular version of CUT is the most difficult to analyze. We also cannot find any pattern here. In [3], it was shown that this game is equivalent to the take-and-break game with hexadecimal code 0.7F.

Other problems from DDLP

Open Problem 2. Is it true that every game $G(\mathcal{C})$ with $\mathcal{C} \neq \{1, 2\}$ has a Sprague-Grundy sequence that is either ultimately periodic or ultimately arithmetic-periodic?

A first step to understand this problem would be to justify (or disprove) the following conjecture.

Conjecture 1. Every instance $\mathcal{C}$ of cut for which $\{1, 2\} \not\subset \mathcal{C}$ is (ultimately) arithmetic-periodic.

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