Bijective Proofs and Integer Partition

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1 Introduction

Combinatorial argument to show summation identities:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}.$$

The statements in each problem are to be proved combinatorially, in most cases by exhibiting an explicit bijection between two sets.

Bijective proof is one of the technique in combinatorics to show the identity or showing that the two sets have the same cardinality. In this talk I will focus on applying this technique to the field called *Partition Numbers*.

2 Integer Partitions

An integer partition is a way of writing n as a sum of positive integers. The number of integer partitions of n is given by the *partition function* $p(n)$. The statement of the flavor “Every number has as many integer partitions of this sort as of that sort” are called *partition identities.*

2.1 Biggest part VS # of parts

The simplest partition identity is the following:

$$p(n|\text{the biggest part is } k) = p(n|\text{exactly } k \text{ parts}).$$

2.2 Odd VS distinct

Goldbach in his letter to Euler in 1742, asked two questions.

1. If any even number greater than 2 could be written as the sum of two prime numbers.

2. The partition identity question:

$$p(n|\text{odd parts}) = p(n|\text{distinct parts}).$$

The first one still open until these days and known as *Goldbach's conjecture*. However Euler solved the second question and sent the solution back within a week.

2.2.1 Generating Functions

Euler's solution use the generating function technique. Here is some preliminary before we get into the proof.

$$\begin{aligned}
\sum_{n\geq 0} p(n)q^n &= (1+q+q^{1+1}+q^{1+1+1}+\dots)(1+q^2+q^{2+2}+q^{2+2+2}+\dots)\dots \\
&= \prod_{i=1}^{\infty}(1+q^i+q^{2i}+q^{3i}+\dots) \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-q^i}\right).
\end{aligned}$$

$$\begin{aligned}
\sum_{n\geq 0} p(n|\text{odd parts})q^n &= (1+q+q^{1+1}+q^{1+1+1}+\dots)(1+q^3+q^{3+3}+q^{3+3+3}+\dots)\dots \\
&= \prod_{i=1}^{\infty}\left(\frac{1}{1-q^{2i-1}}\right).
\end{aligned}$$

$$\begin{aligned}
\sum_{n\geq 0} p(n|\text{distinct parts})q^n &= (1+q^1)(1+q^2)(1+q^3)(1+q^4)(1+q^5)\cdots \\
&= \prod_{i=1}^{\infty}(1+q^i).
\end{aligned}$$

Now the first proof of the identity by Euler in 1742.

Proof.

$$\begin{aligned}
\sum_{n\geq 0} p(n|\text{distinct parts})q^n &= \prod_{i=1}^{\infty}(1+q^i) \\
&= \prod_{i=1}^{\infty}\frac{(1+q^i)(1-q^i)}{(1-q^i)} \\
&= \prod_{i=1}^{\infty}\frac{(1-q^{2i})}{(1-q^i)} \\
&= \prod_{i=1}^{\infty}\frac{1}{(1-q^{2i-1})} \\
&= \sum_{n\geq 0} p(n|\text{odd parts})q^n.
\end{aligned}$$

□

2.2.2 Two Bijective Proofs

A bijection must have the property that when we feed it a collection of odd parts, it delivers a collection of distinct parts with the same sum. Its inverse, of course, must do converse.

First Proof. From odd to distinct parts: merge the largest possible 2^k repeated odd parts. i.e.

$$\begin{aligned} 3+3+3+1+1+1+1 &\mapsto (3+3)+3+(1+1+1+1) \\ &\mapsto 6+3+4 \end{aligned}$$

From distinct to odd parts: split the even part into two equal parts. Repeating the process until the parts are all odd.

$$\begin{aligned} 6+3+4 &\mapsto (3+3)+3+(2+2) \\ &\mapsto 3+3+3+(1+1)+(1+1) \\ &\mapsto 3+3+3+1+1+1+1 \end{aligned}$$

It can be checked that the process is well defined and forms a bijection. □

Second Proof, Sylvester(1884). From odd to distinct parts: lining up the rows of dot representing each part in the center. With the odd parts, we can clearly represent them this way. We then take this representation and associate the dot in a new way (up-right and up-left).

$$13+13+13+11+9+3+3+1 \mapsto 14+12+11+8+7+6+4+3+1.$$

From distinct to odd parts: the inverse mapping appears at first to be somewhat tricky, but it is fairly easily constructed. □

2.3 Other Partition Identities

Since then, many of partition identities has been found. I have a total of 49 identities in my computer. I listed some of them here. It could possibly be a project for a student to find the ways to show these identities.

1. $p(n|\text{even number of odd parts}) = p(n|\text{distinct part, number of odd parts is even})$
2. $p(n|\text{no part divisible by } k) = p(n|\text{less than } k \text{ copies of each part})$
3. $p(n|\#\text{part even}) - p(n|\#\text{part odd}) = (-1)^n p(n|\text{odd dist. part})$
4. $p(n|\text{ parts} \equiv \pm 1 \bmod 3) = p(n|\text{ parts appear at most twice})$
5. $p(n|\text{only odd parts allow to repeat}) = p(n-3|\text{parts appears at most 3 times})$

6. Schur's partition theorem:
   $p(n|\text{parts in } \{1{,}5\} \bmod 6) = p(n|\text{ 3-distinct parts}) = p(n|\text{ distinct part, no part being a multiple of 3})$

7. First Rogers-Ramanujan identity:
   $p(n|\text{parts in } \{1{,}4\} \bmod 5) = p(n|\text{ 2-distinct})$

8. Second Rogers-Ramanujan identity:
   $p(n|\text{parts in } \{2{,}3\} \bmod 5) = p(n|\text{ 2-distinct, parts} \geq 2)$

It is also important to note that the first bijective proof of Rogers-Ramanujan identities was found by Adriano Garsia and Stephen Milne in 1981. But their bijections are very complicated. No simple bijection of these identities has been found so far.

2.4 Euler's Pentagonal Number and Recurrence of $p(n)$

By playing around with the product $\prod(1-q^n)$.

$$
\begin{aligned}
(1-q)(1-q^2)(1-q^3)(1-q^4)\cdots =& 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-q^{35} \\
& -q^{40}+q^{51}+q^{57}-q^{70}-q^{77}+q^{92}+q^{100}-q^{117}-q^{126} \\
& +q^{145}+q^{155}+\ldots.
\end{aligned}
$$

Euler noticed the pattern and turned it into a theorem. His proof used analytic method.

Theorem 1 (Euler's Pentagonal Number Theorem)**.**

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{j=1}^{\infty}(-1)^j(q^{\frac{3j^2-j}{2}}+q^{\frac{3j^2+j}{2}}).$$

It is not until 1881 when the first bijective proof of Euler's Pentagonal theorem was found.

Theorem 2 (Franklin 1881)**.**

$$p(n|\text{ even \#of distinct parts}) - p(n|\text{odd \#of distinct parts}) =$$

$$
\begin{cases}
1 & \text{if } n = \dfrac{3j^2 \pm j}{2} \text{ for some even integer } j \\
-1 & \text{if } n = \dfrac{3j^2 \pm j}{2} \text{ for some odd integer } j \\
0 & \text{otherwise}
\end{cases}
$$

Proof. Swap between the smallest part and the 45 degree part of each element in the Ferrers graphs. □

Example: $n = 12$

Even number of distinct parts	Odd number of distinct parts
11+1	12
10+2	9+2+1
9+3	8+3+1
8+4	7+4+1
7+5	6+5+1
6+3+2+1	7+3+2
5+4+2+1	6+4+2
	5+4+3

This bijection theorem implies Euler's Theorem as the left hand side of the statement corresponds to the generating function $\prod_{n=1}^{\infty}(1-q^n)$ and the right hand side corresponds to $1+\sum_{n=1}^{\infty}(-1)^j(q^{\frac{3j^2-j}{2}}+q^{\frac{3j^2+j}{2}})$.

Euler's theorem is particularly important as it gives a recurrence relation of $p(n)$.

Theorem 3 (Recurrence Relation of $p(n)$)**.**

$$\begin{aligned} p(n) &= p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-\dots \\ &= \sum_{j\geq 1}(-1)^{j+1}\left[p\left(n-\frac{3j^2-j}{2}\right)+p\left(n-\frac{3j^2+j}{2}\right)\right]. \end{aligned}$$

Proof. From Euler's Pentagonal Theorem:

$$\begin{aligned} 1 &= \frac{1}{\prod_{i=1}^{\infty}(1-q^i)}\left(1+\sum_{j=1}^{\infty}(-1)^j(q^{\frac{3j^2-j}{2}}+q^{\frac{3j^2+j}{2}})\right) \\ &= \left(\sum_{i\geq 0}p(i)q^i\right)\left(1+\sum_{j=1}^{\infty}(-1)^j(q^{\frac{3j^2-j}{2}}+q^{\frac{3j^2+j}{2}})\right) \end{aligned}$$

By comparing the coefficient of q^n on both sides (which are all 0 for $n \geq 1$), the theorem follows. $\square$

3 Congruences for $p(n)$ and Crank

Major Percy Alexander MacMahon made up a table of values of $p(n), 1 \leq n \leq 200$ using the recurrence relation above, $p(200) = 3972999029388$. To make the table readable, he grouped the entries in blocks of five in the following manner:

n	$p(n)$	n	$p(n)$	n	$p(n)$
0	1	10	42	20	627
1	1	11	56	21	792
2	2	12	77	22	1002
3	3	13	101	23	1255
4	5	14	135	24	1575
5	7	15	176	25	1958
6	11	16	231	26	2436
7	15	17	297	27	3010
8	22	18	385	28	3718
9	30	19	490	29	4565

Ramanujan noticed something very important. The last $p(n)$ entry in each block is divisible by 5. So he conjectured

$$p(5n+4) \equiv 0 \mod 5.$$

Very soon he added the following conjectures:

$$p(7n+5) \equiv 0 \mod 7.$$

$$p(11n+6) \equiv 0 \mod 11.$$

The first two are easy to prove. The proofs by G.H. Hardy from his book "Ramanujan" (1940) page 87-88 go as follows:

First recall the identities of Euler and Jacobi:

$$E(q) = \prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{n=1}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}(1+q^n), \qquad \text{and}$$

$$E(q)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}.$$

Also recall the obvious fact that follows from the binomial theorem and Fermat's little theorem, that for any prime p and any polynomial or formal power series, $f(q)^p \equiv f(q^p) \mod p$. In particular $E(q)^p \equiv E(q^p) \mod p$.

$p(5n+4)$ ***is divisible by 5***. Since $\left\{\dfrac{n(n+1)}{2} \mod 5;\ 0 \le n \le 4,\ 2n+1 \not\equiv 0 \mod 5\right\} = \{0,1\}$, we have

$$E(q)^3 \equiv J_0 + J_1 \mod 5$$

where J_i consists of those terms in which the power of q is congruent to i modulo 5. Now

$$\sum_{n=0}^{\infty} p(n)q^n = E(q)^{-1} = \frac{(E(q)^3)^3}{(E(q)^5)^2} = \frac{(J_0+J_1)^3}{(E(q)^5)^2} \mod 5.$$

Since $(J_0+J_1)^3 = J_0^3 + 3J_0^2J_1 + 3J_0J_1^2 + J_1^3$, whose terms consist of powers of q that are 0,1,2,3 modulo 5. Therefore none of the powers of q are congruent to 4 mod 5, and hence the coefficient of q^{5n+4} is always 0 modulo 5. $\square$

$p(7n+5)$ ***is divisible by 7.*** Since $\left\{\dfrac{n(n+1)}{2} \mod 7;\ 0 \le n \le 6,\ 2n+1 \not\equiv 0 \mod 7\right\} = \{0,1,3\}$, we have

$$E(q)^3 \equiv J_0 + J_1 + J_3 \mod 7$$

where J_i consists of those terms in which the power of q is congruent to i modulo 7. Now

$$\sum_{n=0}^{\infty} p(n)q^n = E(q)^{-1} = \frac{(E(q)^3)^2}{E(q)^7} = \frac{(J_0+J_1+J_3)^2}{E(q)^7} \mod 7.$$

Since $(J_0+J_1+J_3)^2 = J_0^2 + J_1^2 + J_3^2 + 2J_0J_1 + 2J_0J_3 + 2J_1J_3$, whose terms consist of powers of q that are 0,2,6,1,3,4 modulo 7. Therefore none of the powers of q are congruent to 5 mod 7, and hence the coefficient of q^{7n+5} is always 0 modulo 7. $\square$

Note that the proof of $p(11n+6) \equiv 0 \mod 11$ is harder.

3.1 Dyson's Rank and Crank

Still even in here, people still find a better way to explain Ramanujan's congruences. In 1944, Freeman Dyson defined the rank function to explain the congruences mod 5 and mod 7. In his own word:

"I gave thanks to Ramanujan for two things, for discovering congruence properties of partitions and for not discovering the criterion for dividing them into equal classes."

Definition (Rank)**.** The rank of a partition is the largest part minus the number of parts.

Definition. Let $N(m,t,n)$ denote the number of partitions of n of rank congruent to m modulo t.

The followings were conjectured by Dyson and proved by Atkin and Swinnerton-Dyer in 1954.

$$N(m,5,5n+4) = \frac{1}{5}p(5n+4), \qquad 0 \le m \le 4;$$
$$N(m,7,7n+5) = \frac{1}{7}p(7n+5), \qquad 0 \le m \le 6;$$

Example: Partitions of the integer 9

Partition with rank $\equiv 0 \mod 5$	Partition with rank $\equiv 1 \mod 5$	Partition with rank $\equiv 2 \mod 5$	Partition with rank $\equiv 3 \mod 5$	Partition with rank $\equiv 4 \mod 5$
7+2	8+1	6+1+1+1	9	7+1+1
5+1+1+1+1	5+2+1+1	5+3+1	6+2+1	6+3
4+3+1+1	4+4+1	5+2+2	5+4	4+2+1+1+1
4+2+2+1	4+3+2	3+2+1+1+1+1	3+3+1+1+1	3+3+2+1
3+3+3	$3+1^6$	2+2+2+2+1	4+1+1+1+1+1	3+2+2+2
$2+2+1^5$	$2+2+2+1^3$	1^9	3+2+2+1+1	$2+1^7$

However the *rank function* does not work for mod 11. Dyson conjectured the existence of a *crank function* for partitions that would provide a **combinatorial proof** of Ramanujan's congruences modulo 11.

"Whatever these guesses are warranted by evidence, I leave to the reader to decide. Whatever the final verdict may be, I believe the crank is unique among arithmetic functions in having been named before it was discovered."

He was correct. Forty years later, George Andrews and Frank Garvan (1988) successfully found such a function, and proved the celebrated result that the crank simultaneously explains the three Ramanujan congruences modulo 5, 7 and 11.

Definition (Crank)**.** For a partition π, let $l(\pi)$ be the largest part of π, $\omega(\pi)$ be the number ones in π and $\mu(\pi)$ be the number of parts of π larger than $\omega(\pi)$.
The crank $c(\pi)$ is given by

$$c(\pi) = \begin{cases} l(\pi) & \text{if } \omega(\pi) = 0, \\ \mu(\pi) - \omega(\pi) & \text{if } \omega(\pi) > 0. \end{cases}$$

Definition. Let $N_V(c, t, n)$ denote the number of partitions of n of crank congruent to c modulo t.

This function was successfully divides Ramanujan's congruences equally.

$$\begin{aligned} N_V(c, 5, 5n+4) &= \frac{1}{5} p(5n+4), \qquad 0 \le m \le 4; \\ N_V(c, 7, 7n+5) &= \frac{1}{7} p(7n+5), \qquad 0 \le m \le 6; \\ N_V(c, 11, 11n+6) &= \frac{1}{11} p(11n+6), \qquad 0 \le m \le 10. \end{aligned}$$

However in their paper, Andrews and Garvan proved these formulas analytically. It was Dyson himself who gave a combinatorial proof one year later. (This could be another interesting project for an undergrad. student as well.)

Example: Partitions of the integer 6

Partition of 6	$l(\pi)$	$\mu(\pi)$	$\omega(\pi)$	$c(\pi)$
6	6	1	0	6
5+1	5	1	1	0
4+2	4	2	0	4
4+1+1	4	1	2	-1
3+3	3	2	0	3
3+2+1	3	2	1	1
3+1+1+1	3	0	3	-3
2+2+2	2	3	0	2
2+2+1+1	2	0	2	-2
2+1+1+1+1	2	0	4	-4
1+1+1+1+1+1	2	0	6	-6

3.2 Epilogue

It can also be shown that there is no congruence of the form $p(bk+a) \equiv 0 \pmod{b}$ for any prime b other than 5, 7, or 11. In the 1960s, A. O. L. Atkin of the University of Illinois at Chicago discovered additional congruences for small prime moduli. For example:

$$p(11^3 \cdot 13 \cdot k + 237) \equiv 0 \pmod{13}.$$

Ono (2000) proved that there are such congruences for every prime modulus. Later, Ahlgren & Ono (2001) showed there are partition congruences modulo every integer coprime to 6. For example, his results give

$$p(4063467631k + 30064597) \equiv 0 \pmod{31}.$$

4 Other Interesting Identities

There are still many pretty bijective proofs that I do not have time to cover during this talk, i.e.

Theorem 4 (Cayley's formula)**.** *For every positive integer $n, n \geq 2$, the number of trees on n labeled vertices is n^{n-2}.*

Theorem 5 ($n!$=(YT,YT))**.** *The number of elements in S_n is equal to the number of pairs of standard tableaux of the same shape λ as λ varies over all partitions of n.*

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

Theorem 6 (Hook Length Formula)**.** *Let d_λ be the number of standard Young tableaux of shape λ. Then*

$$d_\lambda = \frac{n!}{\prod h_\lambda(i,j)}.$$

Appendix of Partition Numbers

An Upper Bound for $p(n)$

Theorem 7. $p(n) \leq F_{n+1}, \quad n \geq 0.$

Proof. We know $p(n) = p(n-1) + p(n|$ no 1-part).
(Note: $p(n-2) = p(n|$ at least one 2-part).)
and $p(n-2) = p(n|$ no 1-part$) + p(n-2|$ smallest non-1-part $< 2+ \#$ 1-parts).
(need some work here)

Therefore $p(n) = p(n-1) + p(n-2) - p(n-2|$ smallest non-1-part $< 2+ \#$ 1-parts).

$$\leq p(n-1) + p(n-2) \quad \text{for } n \geq 2$$

Since $p(1) = 1$ and $p(2) = 2$, we use induction to conclude that
$p(n) \leq F_{n+1}$ for all $n \geq 0$. □

In fact $\lim_{n\to\infty} F_n^{\frac{1}{n}} = \frac{1+\sqrt{5}}{2}$,while $\lim_{n\to\infty} p(n)^{\frac{1}{n}} = 1$.

Jacobi's Triple Product, $\prod_{n=1}^{\infty}(1-q^n)$ and $\prod_{n=1}^{\infty}(1-q^n)^3$

Theorem 8 (Jacobi's triple product identity)**.**

$$\sum_{n=-\infty}^{\infty} z^n q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) \qquad \textit{for } |q|<1, z \neq 0.$$

Proof. Let $J(z) = \prod_{n=1}^{\infty}(1+zq^n)(1+z^{-1}q^{n-1})$

We may expand $J(z)$ in a Laurent series around $z=0$, so

$$J(z) = \sum_{n=-\infty}^{\infty} A_n(q) z^n \tag{1}$$

Furthermore,

$$
\begin{aligned}
J(zq) &= \prod_{n=1}^{\infty}(1+zq^{n+1})(1+z^{-1}q^{n-2}) \\
&= \frac{1+z^{-1}q^{-1}}{1+zq}\cdot J(z) \\
&= z^{-1}q^{-1}\cdot J(z)
\end{aligned}
$$

On the other, from (1)

$$J(zq) = \sum_{n=-\infty}^{\infty} A_n(q)q^n z^n \tag{2}$$

Comparing coeff. of z^n in both $J(zq)$ to get

$$
\begin{aligned}
A_n(q)q^{n+1} &= A_{n+1}(q) \\
\text{or } A_n(q) &= q^n A_{n-1}(q) \\
\text{Therefore } A_n(q) &= q^{\frac{n(n+1)}{2}} A_0(q)
\end{aligned}
$$

Note that $A_0(q)$ is a constant term of $\prod_{n=1}^{\infty}(1+zq^n)(1+z^{-1}q^{n-1})$.

There is a combinatorial interpretation (using Dufree Square) that

$$A_0(q) = \sum_{j=0}^{\infty} p(j)q^j = \prod_{j=1}^{\infty}\frac{1}{1-q^j}.$$

Therefore $\displaystyle\prod_{n=1}^{\infty}(1+zq^n)(1+z^{-1}q^{n-1}) = \left(\prod_{j=1}^{\infty}\frac{1}{1-q^j}\right)\cdot\sum_{n=-\infty}^{\infty} q^{\frac{n(n+1)}{2}}z^n.$

Then the proof is done. □

Corollary 9 (Euler's Pentagonal Number)**.**

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{n=1}^{\infty}(-1)^n q^{\frac{n(3n-1)}{2}}(1+q^n).$$

Proof. In Jocobi's triple product identity, substitute $q \to q^3$ and $z \to -q^{-1}$ to get

$$
\begin{aligned}
\prod_{n=1}^{\infty}(1-q^{3n})(1-q^{3n-1})(1-q^{3n-2}) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} \\
\prod_{n=1}^{\infty}(1-q^n) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}
\end{aligned}
$$

And the statement follows. □

Corollary 10 (Jacobi)**.**

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}.$$

Proof. First we rearrange the term on the left hand side of Jacobi's Triple Product.

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} z^n q^{\frac{n(n+1)}{2}} &= \sum_{n=0}^{\infty} z^n q^{\frac{n(n+1)}{2}} + \sum_{n=-\infty}^{-1} z^n q^{\frac{n(n+1)}{2}} \\
&= \sum_{n=0}^{\infty} z^n q^{\frac{n(n+1)}{2}} + \sum_{n=0}^{\infty} z^{-n-1} q^{\frac{n(n+1)}{2}} \\
&= \sum_{n=0}^{\infty} \left(\frac{z^{2n+1}+1}{z^{n+1}}\right) q^{\frac{n(n+1)}{2}}
\end{aligned}$$

The right hand side can be written as $\left(1+\dfrac{1}{z}\right)\prod_{n=1}^{\infty}(1-q^n)(1+zq^n)(1+z^{-1}q^n)$.

Now divides both sides by $1+\frac{1}{z}$ and take limit $z \to -1$ to get the result. $\square$