THREE RESULTS OF COMBINATORIAL GAME TOADS AND FROGS

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ABSTRACT. We prove values of the starting positions $\mathrm{T}^a\square\square\square\mathrm{F}^b$, $\mathrm{T}^a\square\square\mathrm{F}^b$ and $\mathrm{T}^a\square\square\square\mathrm{FFF}$. The last two positions were Erickson's conjectures.

1. INTRODUCTION:

The game *Toads and Frogs*, invented by Richard Guy, is extensively discussed in "Winning Ways"[1], the famous classic by Elwyn Berlekemp, John Conway, and Richard Guy, that is the *bible* of combinatorial game theory. This game got so much coverage because of the simplicity and elegance of its rules, the beauty of its analysis, and as an example of a combinatorial game whose positions do not always have values that are numbers.

Rule

The game is played on a $1 \times n$ strip with either Toad(T) , Frog(F) or $\square$ on the squares. Left plays T and Right plays F. T may move to the immediate square on its right, if it happens to be empty, and F moves to the next empty square on the left, if it is empty. If T and F are next to each other, they have an option to jump over one another, in their designated directions, provided they land on an empty square. (see [1] page 14).

In symbols: the following moves are legal for Toad:

$$\ldots\mathrm{T}\square\ldots \quad\rightarrow\quad \ldots\square\mathrm{T}\ldots,$$
$$\ldots\mathrm{TF}\square\ldots \quad\rightarrow\quad \ldots\square\mathrm{FT}\ldots \quad ,$$

and the following moves are legal for Frog:

$$\ldots\square\mathrm{F}\ldots \quad\rightarrow\quad \ldots\mathrm{F}\square\ldots,$$
$$\ldots\square\mathrm{TF}\ldots \quad\rightarrow\quad \ldots\mathrm{FT}\square\ldots.$$

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Two players take turn moving their pieces. The first player who runs out of move loses. Throughout the paper, we will use the notation X^n to denote n contiguous copies of the Toads and Frogs position X. For example, $\square^3$(TF)^{2}F is shorthand for □□□TFTFF.

To be able to understand the present article, reader need a minimum knowledge of combinatorial game theory, that can be found in [1]. In particular, readers should be familiar with the notions of *value* of a game and sum games which are the bypass reversible move rule, dominated options rule and inequality of two games. (see [1] page 33, 62-64).

The only notations we use are $*(=\{0 \mid 0\})$ and $n*(=\{n \mid n\})$. We will not use any shorthand notation like $\uparrow$, $\Uparrow$, etc.

Background Story

Already in [1] there is some analysis of Toads and Frogs positions, such as TT□FF□ and $\square^a$TF$\square^b$. In 1996, Erickson[2] analyzed more general positions such as T$\square^a$F for any a. At the end he made five conjectures about the values of some families of positions. All of them are starting positions (positions where all T are leftmost and all F are rightmost). Erickson's conjectures were:

E1: T^a□□F^b $= \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$ for all $a > b \geq 2$.
E2: T^a□□□FF $= \{a-2 \mid a-2\}$ for all $a \geq 2$.
E3: T^a□□□FFF $= a - \frac{7}{2}$ for all $a \geq 5$.
E4: T$^a\square^a$F^{a-1} $= 1$ or $\{1 \mid 1\}$ for all $a \geq 1$.
E5: T$^a\square^b$F^a is an infinitesimal for all a, b except (a, b) =(3,2).
E6: Toads and Frogs is NP-hard.

Jesse Hull proved E6 in 2000 (see [3]). Doron Zeilberger and the author proved E2 in [4]. The counterexample to conjecture E4 are in [5]. The results of this paper include the proofs of conjectures E1 and E3. Conjecture E5 is still open.

Recently, in [4], Doron Zeilberger and the author discussed the new algorithm called *the symbolic finite-state method.* This method is an automated algorithm to prove the values of all positions in the given class. For example, T^a□T^{b}F and T^{a}FT^b□ are all the positions in the class □F. We first conjecture values of positions in this class. Then we prove these values all at once by using induction. In practice, the following are the classes that we find the values: □F, □□F, □□□F, □□□□F, □□□□□F, □FF, □□FF, □□□FF, □FFF, TF. All the results above are in [6].

Unfortunately, the symbolic finite-state method does not work with the positions where there are variables on both T and F or both $\square$ and F such as $\mathrm{T}^a\square\square\mathrm{F}^b$. However some of these positions have nice values. In [5], the author categorized all positions with one $\square$ and proved that the value of $\mathrm{T}^a\square\square\mathrm{F}^a$ is an infinitesimal for $a \geq 4$. At the end, we made five conjectures related to these types of positions.

Current Work

The values of the starting positions are of a great interest. Some of them have been investigated in Erickson's paper, [2]. The starting position with the variables on both T and F interested the author a lot. In this paper, we show the values of the three starting positions namely,

$$\mathrm{T}^a\square\square\square\mathrm{T}^b = \{a-b \mid a-b\}, \quad a \geq 4, b \geq 4. \tag{1.1}$$

$$\mathrm{T}^a\square\square\mathrm{F}^b = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \quad a > b \geq 2. \tag{1.2}$$

$$\mathrm{T}^a\square\square\square\mathrm{FFF} = a - \frac{7}{2}, \quad a \geq 5. \tag{1.3}$$

Although these positions have beautiful values, the proofs however seem tedious. The techniques used to prove these positions are similar. The author decides to move the proofs of $\mathrm{T}^a\square\square\mathrm{F}^b$ and $\mathrm{T}^a\square\square\square\mathrm{FFF}$ to the appendices to make the paper more readable.

In Section 3, we show the value of $\mathrm{T}^a\square\square\square\mathrm{T}^b, a \geq 4, b \geq 4$. These positions are important starting positions which cannot be proved by the symbolic finite-state method. The proof is the shortest amongst the three positions.

In the appendix A, we discuss the values of the position $\mathrm{T}^a\square\square\mathrm{F}^b, a > b \geq 2$. The value of this position is the first conjecture of Jeff Erickson in [2]. The proof is not long but tricky. Regarding to this type of position, the values of $\mathrm{T}^a\square\square\square\square\mathrm{T}^b$ and $\mathrm{T}^a\square\square\square\square\square\mathrm{T}^b$ are still unknown.

In the appendix B, we show the value of $\mathrm{T}^a\square\square\square\mathrm{FFF}, a \geq 5$. In theory, we can apply the symbolic finite-state method to prove the values of all positions in the class $\square\square\square\mathrm{FFF}$. The result above follows as one of the special case of many positions in this class. But as we mentioned in [4], we could not get the computer to conjecture all the values of all the positions in this class yet. And it would takes days for human to do conjectures by hand. For now we prove the value of $\mathrm{T}^a\square\square\square\mathrm{FFF}$, $a \geq 5$, which is Erickson's conjecture 3, by hand. The proof is however assisted by computer program the author wrote in Maple.

We do not have an automated algorithm for position with variable on both T and F yet. The proofs are assisted by the old program in making conjectures and doing computation. In the future, we hope to have a new method (hopefully along the same line as the symbolic finite-state method) or computer program to make the proofs more automatic or at least shorten them.

2. Convention and Lemma.

In this section, we explain the notation we use in this paper and also mention two lemmas which we will refer to quite often through out this paper.

2.1. **Convention.** For $G \leq 0$, we want to show that Right can win when Left moves first. We show that for each of the possible Left choices, Right has a response that wins the game.

Similarly, we show $G \leq H$ by considering $G - H \leq 0$. Therefore we want to show that for all the possible Left moves in the game G or Right moves in the game H, there is a winning reply by Right in the game G or Left in the game H.

Below is an example of the notation we use in this paper.

Example: To show: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{FT}\square\mathrm{F}^b \leq \frac{1}{2}, \quad k \geq 0, a \geq 0, b \geq 1.$

$\overset{\overset{2}{\rightarrow}}{\mathrm{T}^a}\ \mathrm{F}\square\mathrm{T}^k\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}^b \leq \overset{\overset{3}{\leftarrow}}{\tfrac{1}{2}}$.

Left has three choices to play, moving T on the left hand side or making a move on the right hand side. The arrows above show the possible moves of Left on the left hand side and Right on the right hand side in an order from 1 to 3. Right could respond to each one of Left's moves as follows:

First choice: Right responds by moving in the left option of $\{0 \mid 1\}$ on the right hand side. This leads to the position:

Case 1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \leq 0.$

Second choice: Right responds by moving the left most F. This leads to the position:

Case 2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\mathrm{FT}\square\mathrm{F}^b \leq \frac{1}{2}.$

Third choice: Left picks Right option of $\{0 \mid 1\}$ on the right hand side. Right responds by moving the right most F. This leads to the position:

Case 3: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\mathrm{FTF}\square\mathrm{F}^{b-1} \leq 1.$

In conclusion, if Left moves as in $\overset{\overset{i}{\rightarrow}}{\cdot}$ it suffices to show Case i.

2.2. **Two Lemmas.** The proof of Lemma 2.1 is similar to the proof of DLP [1, p.127].

Lemma 2.1. ***One side Death Leap Principle*** *(One side DLP): If X is a position with no two consecutive empty squares and the only possible move for Left is a jump, then $X \leq 0$.*

Proof: In such positions, Left's moves are necessarily jumps and always clear a space for Right to reply. As a result, if Left moves first, Right can always reply. $\square$

Examples: $\mathrm{TTF}\square\mathrm{TTF}\square\mathrm{F} \leq 0$ and $\mathrm{TTTF}\square\mathrm{F}\square\mathrm{TF} \leq 0$.

Lemma 2.2. ***Blocking Rule*** *(BR): let X and Y be positions. The position $XFFY \leq X + Y$.*

Proof: Right can always imitate Left's move on the opposite side of equation. Hence if Left moves first, Right can always reply. $\square$

2.3. **Note.** 1) There are the known positions in [4] or [6] which we will claim results without proving them again. Those are

$$\mathrm{T}^a\square\mathrm{T}^b\mathrm{F}\square = a, \quad a \geq 0, b \geq 1. \tag{2.1}$$

$$\mathrm{T}^a\square\square\mathrm{F} = a - 1, \quad a \geq 1. \tag{2.2}$$

$$\mathrm{T}^a\square\square\mathrm{TF} = a, \quad a \geq 1. \tag{2.3}$$

$$\mathrm{T}\square\mathrm{F}^b = \{0 \mid \{-1 \mid 2 - b\}\}, \quad b \geq 1. \tag{2.4}$$

2) When we consider the choices of move, we omit the move of an integer. By the number avoidance theorem (page 147 in [1]), number is an inferior choice than non-number. Amongst the choices of numbers, moving in an integer is the worst one.

3. $\mathrm{T}^a\square\square\square\mathrm{F}^b$

The values of the starting positions are of a great interest. Some of them have been first investigated in Erickson's paper, [2]. The starting positions with the variables on both Toads and Frogs are even more interesting since they could not be proved by the symbolic finite-state method. In this section we show the value

$\mathrm{T}^a\square\square\square\mathrm{T}^b = \{a - b \mid a - b\}, \quad a \geq 4, b \geq 4.$

The proof here is not long. However it would be interesting to develop computer program to make the proof more automatic in the future. The outline of proof is shown below.

$$\mathrm{T}^{a}\square\square\square\mathrm{F}^{b} = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{FF}\square\mathrm{F}^{b-2} = \{a-b \mid a-b\}.$$

$$\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FFF}\square\mathrm{F}^{b-3} = \{a-b \mid a-b\}.$$

FIGURE 1. $\mathrm{T}^{a}\square\square\square\mathrm{F}^{b} = \{a-b \mid a-b\}$

We prove each of the line above starting from the bottom. The top line is the result we want.

Lemma 3.1. $\mathrm{T}^{a-3}\square\mathrm{T}^{3}\square\mathrm{F}^{3}\square\mathrm{F}^{b-3} = \{a-b \mid a-b\}, \quad a \geq 4, b \geq 4.$

Proof: By symmetry, we only need to show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\ \square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}^{3}}\ \square\mathrm{F}^{3}\square\mathrm{F}^{b-3} \leq \{a-b \overset{\overset{3}{\leftarrow}}{\mid} a-b\}.$

Case 1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\ \square\mathrm{TTF}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{FF}\square\mathrm{F}^{b-3} \leq \{a-b \overset{\overset{3}{\leftarrow}}{\mid} a-b\}.$

Case 1.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TFF}\square\mathrm{F}^{b-3} \leq a-b.$

By BR, the left hand side $\leq \mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{T} + \square\mathrm{F}^{b-3} = (a-3)-(b-3) = a-b.$ Refer to (2.1).

Case 1.2: $\mathrm{T}^{a-4}\square\mathrm{TTTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b.$

By BR, the left hand side $\leq \mathrm{T}^{a-4}\square\mathrm{TTTFT}\square + \square\mathrm{F}^{b-3} = (a-4+1)-(b-3) = a-b.$ Refer to (2.1).

Case 1.3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\ \square\mathrm{TTF}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.

Case 1.3.1: $\mathrm{T}^{a-3}\square\mathrm{TTFF}\square\mathrm{TF}\square\mathrm{F}^{b-3} \leq a-b.$

The left hand side $= \mathrm{T}^{a-3}\square + \square\mathrm{TF}\square\mathrm{F}^{b-3} = (a-3)-(b-3) = a-b.$ Refer to (2.1).

Case 1.3.2: $\mathrm{T}^{a-4}\square\mathrm{TTTFTFF}\square\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $(a-4)-2(b-3)=a-2b+2$.

Case 2: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-4}}\ \square\mathrm{TTT}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 2.1: $\mathrm{T}^{a-4}\square\mathrm{TTTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. This is case 1.2.

Case 2.2: $\mathrm{T}^{a-5}\square\mathrm{T}^5\mathrm{F}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

By BR, the left hand side $\leq \mathrm{T}^{a-5}\square\mathrm{T}^5\mathrm{F}\square + \square\mathrm{F}^{b-3} = (a-5)-(b-3) = a-b-2$. Refer to (2.1).

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTTF}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

By BR, the left hand side is $\leq \mathrm{T}^{a-3}\square\mathrm{TTTF}\square + \square\mathrm{F}^{b-3} = (a-3)-(b-3) = a-b$. Refer to (2.1). $\square$

Lemma 3.2. $\mathrm{T}^{a-2}\square\mathrm{T}^2\square\mathrm{F}^2\square\mathrm{F}^{b-2} = \{a-b \mid a-b\}, \quad a \geq 4, b \geq 4$.

Proof: By symmetry, we only need to show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \square\ \overset{\xrightarrow{1}}{\mathrm{T}^2}\ \square\mathrm{F}^2\square\mathrm{F}^{b-2} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \square\mathrm{TF}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{F}^{b-2} \leq \{a-b \overset{\xleftarrow{3}}{\mid} a-b\}$.

Case 1.1: $\mathrm{T}^{a-2}\square\mathrm{TFFT}\square\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$.

Left has to move the left most T, otherwise Right will move the left most F and block the left most position. Therefore we have
$\mathrm{T}^{a-3}\square\mathrm{TTFFT}\square\square\mathrm{F}^{b-2} \leq a-b$.

The left hand side is $(a-3)-(b-3)=a-b$. Refer to (2.2).

Case 1.2: $\mathrm{T}^{a-3}\square\mathrm{TTFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq \{a-b \mid a-b\}$.

This is Case 1 in Lemma 3.1.

Case 1.3: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \square\mathrm{TF}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 1.3.1: $\mathrm{T}^{a-2}\square\mathrm{TFFT}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.

Left has to move the left most T, otherwise Right will move the left most F and block the left most position. Therefore we have
$\mathrm{T}^{a-3}\square\mathrm{TTFFTF}\square\square\mathrm{F}^{b-3} \leq a-b$.

The left hand side is $(a-3)-(b-3) = a-b$. Refer to (2.3).

Case 1.3.2: $\mathrm{T}^{a-3}\square\mathrm{TTFTF}\square\mathrm{F}\square\mathrm{F}^{b-3} \leq a-b$.
This is Case 1.3 in Lemma 3.1.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq \{a-b \mid a-b\}$, true by Lemma 3.1.

Case 3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\ \square\mathrm{T}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{FFF}\square\mathrm{F}^{b-3} \leq a-b$.

Case 3.1: $\mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$. This is Case 1.3 above.

Case 3.2: $a-b \leq a-b$, by Lemma 3.1. $\square$

Lemma 3.3. $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} = \{a-b \mid a-b\}, \quad a \geq 4, \quad b \geq 4.$

Proof: By symmetry, we only need to show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{a-b \overset{\overset{3}{\leftarrow}}{\mid} a-b\}$.

Case 1: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\square\mathrm{TFF}\square\mathrm{F}^{b-2} \leq \{a-b \overset{\overset{2}{\leftarrow}}{\mid} a-b\}$.

Case 1.1: $\mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{F}\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$.
This is Case 1 of Lemma 3.2.

Case 1.2: $\mathrm{T}^{a-1}\square\square\mathrm{TFFF}\square\mathrm{F}^{b-3} \leq a-b$.

$\Rightarrow \mathrm{T}^{a-2}\square\mathrm{TFT}\square\mathrm{FF}\square\mathrm{F}^{b-3} \leq a-b$.
This is Case 1.3 of Lemma 3.2.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{FF}\square\mathrm{F}^{b-2} \leq \{a-b \mid a-b\}$, true by Lemma 3.2.

Case 3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{FF}\square\mathrm{F}^{b-2} \leq a-b$.

Case 3.1: $\mathrm{T}^{a-1}\square\square\mathrm{TFFF}\square\mathrm{F}^{b-3} \le a-b$. This is Case 1.2 above.

Case 3.2: $a-b \le a-b$, by Lemma 3.2. $\square$

Theorem 3.4. $\mathrm{T}^a\square\square\square\mathrm{F}^b = \{a-b \mid a-b\}, \quad a \ge 4, \quad b \ge 4.$

Proof: By symmetry, we only need to show $\overset{\xrightarrow{1}}{\mathrm{T}^a}\,\square\square\square\mathrm{F}^b \le \{a-\overset{\xleftarrow{2}}{b} \mid a-b\}$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F}^{b-1} \le \{a-b \mid a-b\}$, true by Lemma 3.3.

Case 2: $\mathrm{T}^{a}\square\square\mathrm{F}\square\mathrm{F}^{b-1} \le a-b$.

$\Rightarrow a-b \le a-b$, by Lemma 3.3. $\square$

4. Future Work

Toads and Frogs is beautiful game with many nice patterns everywhere. The works done so far could have been just the tip of an iceberg. The first breakthrough was the computer programming which compute the values on a specific board that led to discovering patterns and conjectures as in [2]. The second breakthrough was in [4] which the automated proof was introduced to verify the values of a specific class. Unfortunately, the positions in this paper are too general for that method. The next breakthrough should be another algorithm that are able to handle these general positions. But that is an ultimate goal. In making a progress, it is important to stress the role of computer program. Computer program will help seeing the bigger picture instead of getting lost in details. We have to wait to see whether anyone could find a better algorithm to automated or at least assisted the proof of the new results. The new algorithm might also as well be useful in other games or other brunches of mathematics.

Erickson made 5 conjectures in [2]. Four of them have already been confirmed, three are positive and one is negative. Conjecture 5 is difficult and still open. The author made five new conjectures in [5]. Two of these conjectures are the refine version of conjecture 5 of Erickson. We believe that proving these five new conjectures could lead to a new technique and a better understanding of this game.

We end this paper by stating conjecture and open problem similar to the positions we proved in this paper.

Conjecture:
$\mathrm{T}^a\square\square\square\square\mathrm{F}^b = \{a-b|a-b\}, \quad a \geq 6, b \geq 6.$

Open Problem:
For a fixed number $i \geq 5$, find the values of $\mathrm{T}^a\square^i\mathrm{F}^b$.

APPENDIX

Appendix A. $\mathrm{T}^a\square\square\mathrm{F}^b$

We show $\mathrm{T}^a\square\square\mathrm{F}^b = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \quad a > b \geq 2.$

This position is the smallest nontrivial starting position with variable on both Toads and Frogs. The values of this position are the first conjecture of Erickson in [2]. We will do the case analysis similar to Section 3. The proof is not long but tricky.

For the case $a > b = 2$, the result is already known in [4] and [6].

For the case $a > b \geq 3$,we will prove 11 lemmas that will lead to the main theorem.

We first show the value of $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1}$. We then conclude the result by showing that $\mathrm{T}^a\square\square\mathrm{F}^b = \mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1}$. Below is how the tree looks like at the beginning.

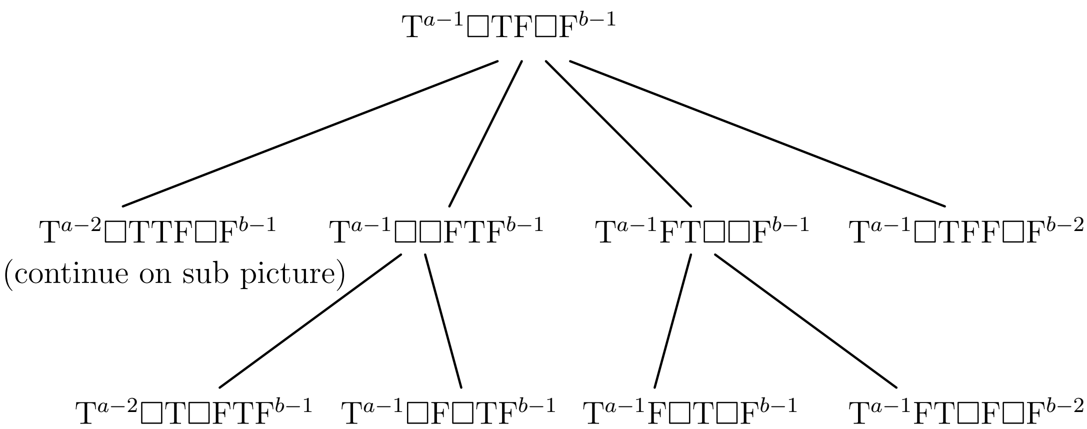

Figure 2. Main tree

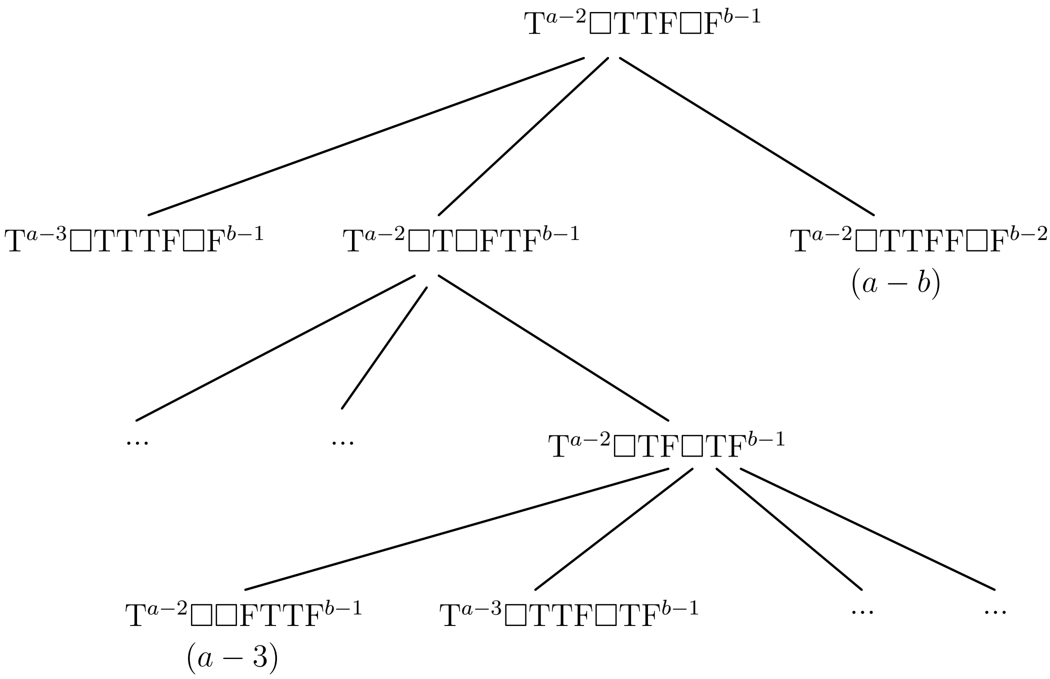

FIGURE 3. Sub picture of the main tree

Lemma A.1,A.2 and A.3 will be useful for the subsequence lemmas.

Lemma A.1. $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b = a, \quad k \geq 2, a \geq 0, b \geq 0.$

Proof: We show

I) $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \leq a, \quad k \geq 2, a \geq 0, b \geq 0.$

II) $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \geq a, \quad k \geq 2, a \geq 0, b \geq 0.$

I) To show $\overset{\xrightarrow{2}}{\mathrm{T}^a}\ \square\ \ \overset{\xrightarrow{1}}{\mathrm{T}^k}\ \mathrm{F}\square\mathrm{TF}^b \leq a, \quad k \geq 2, a \geq 0, b \geq 0.$

Case1: $\mathrm{T}^a\square\ \mathrm{T}^{k-1}\mathrm{F}\square \leq a$. The left hand side is a by (2.1).

Case2: $\mathrm{T}^{a-1}\square\ \mathrm{T}^{k+1}\mathrm{FFT}\square\mathrm{F}^{b-1} \leq a$. This is true by (2.4).

II) To show $\mathrm{T}^a\square\ \mathrm{T}^k\mathrm{F}\square\mathrm{TF}^b \geq a, \quad k \geq 2, a \geq 0, b \geq 0.$

By BR, the left hand side $\geq \mathrm{T}^a\square + \mathrm{F}\square\mathrm{TF}^b = a + 0 = a$. $\square$

Lemma A.2. $\mathrm{T}^a\square\mathrm{T}^k\square\mathrm{FTF}^b \geq a - 1, \quad k \geq 1, a \geq 1, b \geq 0.$

Proof: Left has only choice which leads to: $\mathrm{T}^{a-1}\square\mathrm{T}^{k+1}\mathrm{F}\square\mathrm{TF}^b \geq a - 1$.
This is II) of Lemma A.1. $\square$

Lemma A.3. $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^i\mathrm{F}^j\mathrm{T}\square\mathrm{F}^b \leq \{0 \mid 0\}, \quad i \geq 0, j \geq 1, a \geq 0, b \geq 1.$

Proof: We use induction on a.

Base Case: $a = 0$, $\square\mathrm{T}^i\mathrm{F}^j \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{F}^b \leq \{0 \overset{\xleftarrow{2}}{\mid} 0\}$

Case1: $\square\mathrm{T}^i\mathrm{F}^j\square\mathrm{TF}^b \leq 0$, true by one side DLP.

Case2: $\square\mathrm{T}^i\mathrm{F}^j\mathrm{TF}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP.

Induction Step: $\overset{\xrightarrow{2}}{\mathrm{T}^a} \mathrm{F}\square\mathrm{T}^i\mathrm{F}^j \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{F}^b \leq \{0 \overset{\xleftarrow{3}}{\mid} 0\}, \quad a \geq 1.$

Case1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^i\mathrm{F}^j\square\mathrm{TF}^b \leq 0$, true by one side DLP.

Case2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{i+1}\mathrm{F}^j\mathrm{T}\square\mathrm{F}^b \leq \{0 \mid 0\}$, true by induction.

Case3: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^i\mathrm{F}^j\mathrm{TF}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP. $\square$

Lemma A.4. $\mathrm{T}^a\mathrm{F} \overset{\xrightarrow{2}}{\mathrm{T}} \square\mathrm{F}^k \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{F}^b \leq 0, \quad k \geq 1, a \geq 0, b \geq 2.$

Proof:

Case1: $\mathrm{T}^a\mathrm{F} \overset{\xrightarrow{2}}{\mathrm{T}} \square\mathrm{F}^{k+1} \overset{\xrightarrow{1}}{\mathrm{T}} \square\mathrm{F}^{b-1} \leq 0$

Case1.1: $\mathrm{T}^a\mathrm{FTF}\square\mathrm{F}^k\square\mathrm{TF}^{b-1} \leq 0$, true by one side DLP.

Case1.2: $\mathrm{T}^a\mathrm{F}\square\mathrm{TF}^{k+1}\mathrm{TF}\square\mathrm{F}^{b-2} \leq 0$, true by one side DLP.

Case2: $\mathrm{T}^a\mathrm{F}\square\mathrm{TF}^k\mathrm{TF}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP. $\square$

Lemma A.5. $\mathrm{T}^a\mathrm{F}\overset{\xrightarrow{1}}{\mathrm{T}}\square\mathrm{F}^k\square\mathrm{TF}^b \leq \{0 \overset{\xleftarrow{2}}{|} 0\}, \quad k \geq 0, a \geq 0, b \geq 3.$

Proof:
Case1: $\mathrm{T}^a\mathrm{F}\square\mathrm{TF}^{k+1}\mathrm{T}\square\mathrm{F}^{b-1} \leq \{0 \mid 0\}$, true by Lemma A.3.

Case2: $\mathrm{T}^a\mathrm{FT}\square\mathrm{F}^{k+1}\mathrm{T}\square\mathrm{F}^{b-1} \leq 0$, true by Lemma A.4. $\square$

Lemma A.6 will be used for bypass reversible move rule in figure 3.

Lemma A.6. $\overset{\xrightarrow{2}}{\mathrm{T}^a}\square\overset{\xrightarrow{1}}{\mathrm{T}}\mathrm{F}\square\mathrm{TF}^b \leq \mathrm{T}^a\square\mathrm{TTF}\square\overset{\xleftarrow{3}}{\mathrm{F}^b}, \quad a \geq b \geq 2.$

Proof:
Case1: $\mathrm{T}^a\square\square\mathrm{F} \leq \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$
$\Rightarrow a-1 \leq \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$, refer to (2.2). True by Lemma A.2.

Case2: $\mathrm{T}^{a-1}\square\mathrm{TTF}\square\mathrm{TF}^b \leq \mathrm{T}^a\square\mathrm{T}\square\mathrm{FTF}^b$.
The left hand side is $a-1$ by Lemma A.1.
Then the statement is true by Lemma A.2.

Case3: $\mathrm{T}^a\mathrm{FT}\square\square\mathrm{TF}^b \leq a-b+1$

$\Rightarrow \mathrm{T}^a\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{b-1} \leq a-b+1$, true by Lemma A.3. $\square$

Lemma A.7. $\overset{\xrightarrow{2}}{\mathrm{T}^a}\square\mathrm{TT}\overset{\xrightarrow{1}}{\mathrm{T}}\mathrm{F}\square\mathrm{F}^b \leq a, \quad a \geq 0, b \geq 1.$

Proof:
Case 1: $\mathrm{T}^a\square\mathrm{TTF}\square\mathrm{TF}^b \leq a$, true by Lemma A.1.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{TTTTFF}\square\mathrm{F}^{b-1} \leq a$. The left hand side is $a-b$. $\square$

Lemma A.8 will be used for bypass reversible move rule in figure 2.

Lemma A.8. $\overset{\xrightarrow{2}}{\mathrm{T}^{a-1}}\square\overset{\xrightarrow{1}}{\mathrm{T}}\mathrm{F}\square\mathrm{TF}^b \leq \mathrm{T}^a\square\square\overset{\xleftarrow{3}}{\mathrm{F}}\mathrm{TF}^b, \quad a \geq 2, b \geq 0.$

Proof:

Case 1: $a-2 \leq \mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{FTF}^{b}$, true by Lemma A.2.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\mathrm{TF}^{b} \leq \mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{FTF}^{b}$, the left hand side is $a-2$, by Lemma A.1. Then the statement is true by Lemma A.2.

Case 3: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{TF}^{b} \leq \mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{TF}^{b}$. This is trivially true. $\square$

Lemma A.9. $\mathrm{T}^{a}\square\mathrm{F}\square\mathrm{TF}^{b} \leq 1, \quad a \geq 2, b \geq 2.$

Proof: $\Rightarrow \mathrm{T}^{a-1}\mathrm{FT}\square\square\mathrm{TF}^{b} \leq 1.$
$\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{TFT}\square\mathrm{F}^{b-1} \leq 1$, true by Lemma A.3. $\square$

Lemma A.10. $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\square\mathrm{F}^{b} = \{0 \mid 0\}, \quad k \geq 1, a \geq 3, b \geq 2.$

Proof: Need to show

I) $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\square\mathrm{F}^{b} \leq \{0 \mid 0\}, \quad k \geq 1, a \geq 0, b \geq 2.$

II) $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\square\mathrm{F}^{b} \geq \{0 \mid 0\}, \quad k \geq 1, a \geq 3, b \geq 1.$

I) We use induction on a

Base Case: $a=0$, $\square \overset{\overset{1}{\rightarrow}}{\mathrm{T}^{k}} \square\mathrm{F}^{b} \leq \{0 \overset{\overset{2}{\leftarrow}}{\mid} 0\}$.

Case 1: $\square\mathrm{T}^{k-1}\mathrm{FT}\square\mathrm{F}^{b-1} \leq \{0 \mid 0\}$, true by Lemma A.3.

Case 2: $\square\mathrm{T}^{k}\mathrm{F}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP.

Induction step: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a}} \mathrm{F}\square \overset{\overset{1}{\rightarrow}}{\mathrm{T}^{k}} \square\mathrm{F}^{b} \leq \{0 \overset{\overset{3}{\leftarrow}}{\mid} 0\}, \quad a \geq 1.$

Case 1: $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k-1}\mathrm{FT}\square\mathrm{F}^{b-1} \leq \{0 \mid 0\}$, true by Lemma A.3.

Case 2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^{k+1}\square\mathrm{F}^{b} \leq \{0 \mid 0\}$, true by induction.

Case 3: $\mathrm{T}^{a}\mathrm{F}\square\mathrm{T}^{k}\mathrm{F}\square\mathrm{F}^{b-1} \leq 0$, true by one side DLP.

II) To show $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^k\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}{}^b \geq \{0 \overset{\overset{2}{\rightarrow}}{|}\ 0\}, \quad k \geq 1, a \geq 3, b \geq 1.$

Case 1: $\mathrm{T}^a\mathrm{F}\square\mathrm{T}^{k-1}\square\mathrm{FTF}^{b-1} \geq \{0 \mid 0\}$, true by negative of Lemma A.5.

Case 2: $\mathrm{T}^{a-1}\square\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{T}^{k+1}\square\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}{}^b \geq 0.$

Case 2.1: $\mathrm{T}^{a-1}\square\mathrm{FT}^k\square\mathrm{FTF}^{b-1} \geq 0$, true by negative of Lemma A.4.

Case 2.2: $\mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}^k\square\mathrm{TF}^b \geq 0$, true by one side DLP. $\square$

Lemma A.11. $\{* \mid \overset{\overset{1}{\rightarrow}}{1} - b\} \ \leq \mathrm{T}^a\square\mathrm{T}\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square\ \overset{\overset{3}{\leftarrow}}{\mathrm{F}}{}^b, \quad b \geq 1, a \geq b + 2.$

Proof:

Case 1: $* \leq a - b - 1.$

Case 2: $\{* \mid \overset{\overset{1}{\rightarrow}}{1} - b\} \leq \mathrm{T}^a\mathrm{F}\square\mathrm{T}\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \square\ \overset{\overset{3}{\leftarrow}}{\mathrm{F}}{}^b.$

Case 2.1: $* \leq \mathrm{T}^a\mathrm{F}\square\square\mathrm{FTF}^b$, true by the negative of Lemma A.5.

Case 2.2: $1 - b \leq \mathrm{T}\square\square\mathrm{F}^b$. The right hand side is $1 - b$, by (2.2).

Case 2.3: $1 - b \leq \mathrm{T}^a\mathrm{F}\square\mathrm{TFF}\square\mathrm{F}^{b-1}$, true by the negative of Lemma A.1.

Case 3: $\{* \mid 1 - b\} \leq a - b.$

$\Rightarrow 0 \leq a - b$, which is true. $\square$

After applying Lemma A.1,A.6,A.7,A.8,A.9,A.10,A.11 to the tree in figure 2, it looks like:

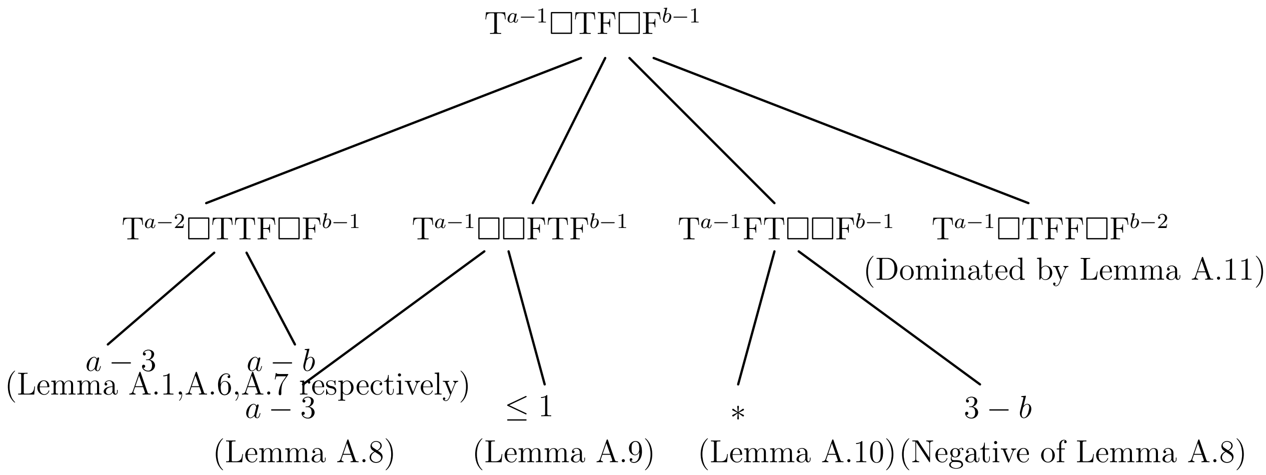

FIGURE 4. Tree after applying Lemma A.1,A.6,A.7,A.8,A.9,A.10,A.11

It might be helpful to provide some explanation of the value $a-3$ after applying Lemma A.8 above. We apply the bypass reversible move by comparing the position $\mathrm{T}^{a-1}\square\square\mathrm{FTF}^{b-1}$ and $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{TF}^{b-1}$ which is the only right option of $\mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{FTF}^{b-1}$. The left options of $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{TF}^{b-1}$ are $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}^{b-1}$ and $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}^{b-1}$ in which both have values $a-3$ by (2.2) and Lemma A.1 respectively.

After applying these lemmas, we finally conclude that
$\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$.

We finally show the main theorem using the result above.

Theorem A.12. $\mathrm{T}^{a}\square\square\mathrm{F}^{b} = \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

Proof: Need to show

I) $\mathrm{T}^{a}\square\square\mathrm{F}^{b} \leq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

II) $\mathrm{T}^{a}\square\square\mathrm{F}^{b} \geq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

I) To show $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}}\,\square\square\mathrm{F}^{b} \leq \{\{a-3 \mid a-\overset{\overset{2}{\leftarrow}}{b}\} \mid \{* \mid 3-b\}\}, \;\; a > b \geq 3.$

Case 1: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} \leq \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, true by the tree above.

Case 2: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}}\,\square\mathrm{F}\square\mathrm{F}^{b-1} \leq \{* \mid 3-\overset{\overset{2}{\leftarrow}}{b}\}$.

Case 2.1: $\{* \mid 3-b\} \le \{* \mid 3-b\}$, true by the tree above.

Case 2.2: $\mathrm{T}^a\square\mathrm{FF}\square\mathrm{F}^{b-2} \le 3-b$.
$\Rightarrow \mathrm{T}^{a-1}\mathrm{FT}\square\mathrm{F}\square\mathrm{F}^{b-2} \le 3-b$, true by the negative of Lemma A.2.

II) To show $\mathrm{T}^a\square\square\, \overset{\overset{1}{\leftarrow}}{\mathrm{F}}{}^b \;\ge\; \{\{a-3 \mid a-\overset{\overset{2}{\rightarrow}}{b}\} \mid \{* \mid 3-b\}\}$, $a > b \ge 3$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{TF}\square\mathrm{F}^{b-1} \ge \{\{a-3 \mid a-b\} \mid \{* \mid 3-b\}\}$, true by the tree above.

Case 2: $\mathrm{T}^{a-1}\square\mathrm{T}\square\, \overset{\overset{1}{\leftarrow}}{\mathrm{F}}{}^b \;\ge\; \{a-3 \overset{\overset{2}{\rightarrow}}{\mid} a-b\}$.

Case 2.1: $\{a-3 \mid a-b\} \ge \{a-3 \mid a-b\}$, true by the tree above.

Case 2.2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square\mathrm{F}^b \ge a-3$. This is the negative of I) Case 2.2. $\square$

Appendix B. $\mathrm{T}^a\square\square\square\mathrm{FFF}$

We show $\mathrm{T}^a\square\square\square\mathrm{FFF} = a - \frac{7}{2}$, $a \ge 5$.

We are supposed to be able to prove the values of the class $\square\square\square\mathrm{FFF}$ by the symbolic finite-state method (see [4],[6]). Then the result above will follow as one of the special case of many positions in the class. However there are too many possible cases to make conjectures. For now we prove the value of $\mathrm{T}^a\square\square\square\mathrm{FFF}$, $a \ge 5$ by hand. The proof is quite lengthy but the plan is clear. We need 7 lemmas before we can prove the main theorem.

Below is the outline.

$$\mathrm{T}^{a}\square\square\square\mathrm{FFF} = a - \tfrac{7}{2}$$

$$\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} = a - \tfrac{7}{2}$$

$$\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} = a - \tfrac{7}{2}$$

$$\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} = a - \tfrac{7}{2}$$

$$\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} = a - \tfrac{7}{2} \qquad \mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} = a - \tfrac{7}{2}$$

$$\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} = a - \tfrac{7}{2}$$

Figure 5. Main tree

We first show Lemma B.1. Then we will start to work from the bottom of the main tree (figure 5) and go all the way up to the top.

Lemma B.1. $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a - \frac{7}{2}$, $a \geq 5$.

Proof: To Show $\overset{\overset{3}{\rightarrow}}{\mathrm{T}^{a-3}} \square \overset{\overset{2}{\rightarrow}}{\mathrm{T}} \mathrm{F}\square \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \mathrm{F}\square\mathrm{TF} \leq \overset{\overset{4}{\leftarrow}}{a} - \frac{7}{2}$.

Case 1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\square\mathrm{FTTF} \leq a - 4$.

The left hand side is $\mathrm{T}^{a-3}\square\mathrm{TF}\square\square\mathrm{F} = a - 4, \quad a \geq 4$. This value is the result in [4] (see also [6]).

Case 2: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}} \square\square\mathrm{FT} \overset{\overset{1}{\rightarrow}}{\mathrm{T}} \mathrm{F}\square\mathrm{TF} \leq a - 4$.

Case 2.1: $\mathrm{T}^{a-3}\square\square\mathrm{FTF}\square \leq a-4$. The left hand side already is $a-4$.

Case 2.2: $\mathrm{T}^{a-4}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{TF} \leq a-4$, true by one side DLP.

Case3: $\overset{\xrightarrow{3}}{\mathrm{T}^{a-4}}\ \square\mathrm{T}\ \overset{\xrightarrow{2}}{\mathrm{T}}\ \mathrm{F}\square\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF} \leq a-4$.

Case 3.1: $\mathrm{T}^{a-4}\square\mathrm{TTF}\square\mathrm{F}\square \leq a-4$. The left hand side is $\{\{a-4 \mid a-4\} \mid a-4\}$.

Case 3.2: $\mathrm{T}^{a-4}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{TF} \leq a-4$, true by one side DLP.

Case 3.3: $\mathrm{T}^{a-5}\square\mathrm{TTTFFT}\square\square\mathrm{TF} \leq a-4$. The left hand side is $(a-5)+1$.

Case 4: $\mathrm{T}^{a-3}\mathrm{F}\ \overset{\xrightarrow{2}}{\mathrm{T}}\ \square\square\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{TF} \leq a-3$.

Case 4.1: $\mathrm{T}^{a-3}\mathrm{FT}\square\square\mathrm{F}\square \leq a-3$. The left hand side is $\{\{a-3 \mid a-3\} \mid a-3\}$.

Case 4.2: $\overset{\xrightarrow{2}}{\mathrm{T}^{a-3}}\ \mathrm{F}\square\mathrm{TF}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \square\square\mathrm{TF} \leq a-3$.

Case 4.2.1: $\mathrm{T}^{a-3}\mathrm{FFT}\square\square\mathrm{T}\square\mathrm{TF} \leq a-3$.
The left hand side is $\{3 \mid 3\}$ which assure the statement for $a \geq 7$.

Case 4.2.2: $\mathrm{T}^{a-4}\mathrm{F}\square\mathrm{TTFT}\square\square\mathrm{TF} \leq a-3$.
$\Rightarrow \mathrm{T}^{a-2}\mathrm{FT}\square\square\mathrm{TF} \leq a-3$, The left hand side is 1.

We now have Lemma B.1. We are one step closer to the main theorem. We now prove the statement at the bottom of the tree. □

Lemma B.2. $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} = a - \frac{7}{2}$, $a \geq 5$.

Proof: Need to show

I) $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{TF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\xrightarrow{3}}{\mathrm{T}^{a-3}}\ \Box\ \overset{\xrightarrow{2}}{\mathrm{T}}\ \mathrm{F}\Box\mathrm{T}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\Box\mathrm{F} \leq\ a \overset{\xleftarrow{4}}{-} \frac{7}{2}$.

Case1: $\mathrm{T}^{a-3}\Box\mathrm{TF}\Box\mathrm{TF}\Box\mathrm{TF} \leq a - \frac{7}{2}$, true by Lemma B.1.

Case 2: $\mathrm{T}^{a-3}\Box\Box\mathrm{FTTTFF}\Box \leq a - \frac{7}{2}$. The left hand side is $a-4$.

Case 3: $\mathrm{T}^{a-4}\Box\mathrm{TTF}\Box \leq a - \frac{7}{2}$. The left hand side is $a-4$.

Case 4: $\mathrm{T}^{a-3}\Box\mathrm{TF}\Box \leq a-3$. The left hand side is $a-3$.

II) To Show $\mathrm{T}^{a-3}\Box\mathrm{T}\ \overset{\xleftarrow{1}}{\mathrm{F}}\ \Box\mathrm{TTF}\Box\ \overset{\xleftarrow{2}}{\mathrm{F}}\ \geq\ a \overset{\xrightarrow{3}}{-} \frac{7}{2}$.

Case1: $\mathrm{T}^{a-3}\mathrm{F}\Box\mathrm{T}\Box\mathrm{TTF}\Box\mathrm{F} \geq a - \frac{7}{2}$. The left hand side is $\geq 1 + (a-3) - 1 = a-3$.

Case 2: $\mathrm{T}^{a-3}\Box\mathrm{TF}\Box \geq a-3$. The left hand side is $a-3$.

Case 3: $\mathrm{T}^{a-3}\Box\mathrm{T}\ \overset{\xleftarrow{1}}{\mathrm{F}}\ \Box\mathrm{T}\Box\ \overset{\xleftarrow{2}}{\mathrm{F}}\ \mathrm{TF} \geq\ a-4$.

Case 3.1: $\mathrm{T}^{a-3}\mathrm{F}\Box\mathrm{T}\Box\mathrm{T}\Box\mathrm{FTF} \geq a-4$.
$\Rightarrow \mathrm{T}^{a-3}\mathrm{F}\Box\Box\mathrm{TTF}\Box\mathrm{TF} \geq a-4$. The left hand side is $\geq a-3$, by BR.

Case 3.2: $\mathrm{T}^{a-3}\Box\mathrm{TF}\Box\Box\mathrm{FTTF} \geq a-4$. The left hand side is $a-4$. $\Box$

Lemma B.3. $\mathrm{T}^{a-2}\Box\Box\mathrm{FTTF}\Box\mathrm{F} = a - \frac{7}{2}$, $a \geq 5$.

Proof: Need to show

I) $\mathrm{T}^{a-2}\Box\Box\mathrm{FTTF}\Box\mathrm{F} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a-2}\Box\Box\mathrm{FTTF}\Box\mathrm{F} \geq a - \frac{7}{2}$.

I) To Show $\overset{\xrightarrow{2}}{\mathrm{T}^{a-2}}\ \Box\Box\mathrm{FT}\ \overset{\xrightarrow{1}}{\mathrm{T}}\ \mathrm{F}\Box\mathrm{F} \leq\ a \overset{\xleftarrow{3}}{-} \frac{7}{2}$.

Case1: $\overset{\xrightarrow{2}}{\mathrm{T}}{}^{a-2}\,\square\square\mathrm{F}\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{TF} \leq \overset{\xleftarrow{3}}{a-}\,\frac{7}{2}$.

Case 1.1: $\mathrm{T}^{a-2}\square\square\mathrm{F}\square\mathrm{F} \leq a-4$. The left hand side is $a-4$.

Case 1.2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a-\frac{7}{2}$, true by Lemma B.1.

Case 1.3: $\overset{\xrightarrow{2}}{\mathrm{T}}{}^{a-2}\,\square\mathrm{F}\square\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{TF} \leq a-3$.

Case 1.3.1: $\mathrm{T}^{a-2}\mathrm{F}\square\square\square\mathrm{F} \leq a-3$. The left hand side is $a-3$.

Case 1.3.2: $\mathrm{T}^{a-3}\mathrm{FT}\square\square\mathrm{TF}\square\mathrm{TF} \leq a-3$. This is case 4 of Lemma B.1.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by Lemma B.2.

Case 3: $\mathrm{T}^{a-2}\square\square\mathrm{FTTFF}\square \leq a-3$. The left hand side is $a-3$.

II) To Show $\mathrm{T}^{a-2}\square\square\,\overset{\xleftarrow{1}}{\mathrm{F}}\,\mathrm{TTF}\square\,\overset{\xleftarrow{2}}{\mathrm{F}} \geq \overset{\xrightarrow{3}}{a-}\,\frac{7}{2}$.

Case1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \geq a-\frac{7}{2}$. The left hand side is $a-\frac{7}{2}$ by Lemma B.2.

Case 2: $\mathrm{T}^{a-2}\square\square\mathrm{F} \geq a-3$. The left hand side is $a-3$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{T}\square\,\overset{\xleftarrow{1}}{\mathrm{F}}\,\mathrm{TTF}\square\,\overset{\xleftarrow{2}}{\mathrm{F}} \geq a-4$.

Case 3.1: $a-4 \geq a-4$, by Lemma B.2.

Case 3.2: $\mathrm{T}^{a-3}\square\square\mathrm{TF} \geq a-4$. The left hand side is $a-3$.

We have Lemma B.3 here. Lemma B.4 is similar to Lemma B.3. They are also at the same level in the picture. □

Lemma B.4. $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} = a-\frac{7}{2}$, $a \geq 5$.

Proof: Need to show

I) $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \geq a - \frac{7}{2}$.

I) To Show $\overset{\overset{3}{\rightarrow}}{\mathrm{T}^{a-3}}\ \square\mathrm{T}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F} \leq\ \overset{\overset{4}{\leftarrow}}{a -}\ \frac{7}{2}$.

Case 1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\ \square\mathrm{T}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F}\square\mathrm{TF} \leq\ \overset{\overset{3}{\leftarrow}}{a -}\ \frac{7}{2}$.

Case 1.1: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{TF} \leq a - \frac{7}{2}$, true by Lemma B.1.

Case 1.2: $\mathrm{T}^{a-4}\square\mathrm{TTTFF}\square\square\mathrm{TF} \leq a - \frac{7}{2}$. The left hand side is $(a-4)-1$.

Case 1.3: $\mathrm{T}^{a-3}\square\mathrm{TTFF}\square\square\mathrm{TF} \leq a - 3$. The left hand side is $(a-3)-1$.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{TF}\square\mathrm{TTF}\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a - \frac{7}{2}$ by Lemma B.2.

Case 3: $\mathrm{T}^{a-4}\square\mathrm{TTTFFT}\square\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a-4$.

Case 4: $\mathrm{T}^{a-3}\square\mathrm{TTFFT}\square\square\mathrm{F} \leq a - 3$. The left hand side is $a-3$.

II) To Show $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{T}\ \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \square\ \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq\ \overset{\overset{3}{\rightarrow}}{a -}\ \frac{7}{2}$.

Case 1: $\mathrm{T}^{a-3}\square\mathrm{TTFFT}\square\square\mathrm{F} \geq a - 3$. The left hand side is $a-3$.

Case 2: $\mathrm{T}^{a-3}\square\mathrm{T}\square\mathrm{FTTFF}\square \geq a - \frac{7}{2}$. The left hand side is $\{a-3 \mid a-3\}$.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{T}\square\mathrm{FTTF}\square\mathrm{F} \quad \geq \quad a - 4$. This is II) Case 3 of Lemma B.3. $\square$

Lemma B.5. $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} = a - \frac{7}{2}$, $a \geq 5$.

Proof: Need to show

I) $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} \leq a - \frac{7}{2}$.

II)) $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\xrightarrow{3}}{\mathrm{T}^{a-2}}\,\square\,\overset{\xrightarrow{2}}{\mathrm{T}}\,\mathrm{F}\square\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\mathrm{FF} \leq\ a \overset{\xleftarrow{4}}{-} \frac{7}{2}$.

Case1: $\mathrm{T}^{a-2}\mathrm{F}\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\square\square\mathrm{TFF} \leq a \overset{\xleftarrow{2}}{-} \frac{7}{2}$.

Case 1.1: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{FT}\square\mathrm{F} \leq a - \frac{7}{2}$.
The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\square\mathrm{FT}\square\mathrm{F} = \{1 \mid 1\}$.

Case 1.2: $\mathrm{T}^{a-2}\mathrm{F}\,\overset{\xrightarrow{2}}{\mathrm{T}}\,\square\square\mathrm{F}\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\square\mathrm{F} \leq a - 3$.

Case 1.2.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{F}\square\square\mathrm{TF} \leq a - 3$.
The left most position will get block eventually.

Case 1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{F} \leq a - 3$.

The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} \leq a - 3$.
The left hand side is $\{\{\{a - 3 \mid 2\} \mid 1\} \mid 0\}$.

Case 2: $\mathrm{T}^{a-2}\square\square\mathrm{FTTF}\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a - \frac{7}{2}$ by Lemma B.3.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \leq a - \frac{7}{2}$. The left hand side is $a - \frac{7}{2}$ by Lemma B.4.

Case 4: $\overset{\xrightarrow{3}}{\mathrm{T}^{a-2}}\,\square\,\overset{\xrightarrow{2}}{\mathrm{T}}\,\mathrm{F}\square\,\overset{\xrightarrow{1}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F} \leq a - 3$.

Case 4.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\square\square\mathrm{FTF} \leq a - 3$.

$\Rightarrow \mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{F}\square\mathrm{TF} \leq a - 3$.

$\Rightarrow \mathrm{T}^{a-1}\square\mathrm{F}\square\mathrm{TF} \leq a - 3$., the left hand side is 1.

Case 4.2: $\mathrm{T}^{a-2}\square\square\mathrm{F} \leq a - 3$. , the left hand side is $a - 3$.

Case 4.3: $a - 3 \leq a - 3$, true by Lemma B.4.

II) To Show $\mathrm{T}^{a-2}\square\mathrm{T}\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\square\mathrm{T}\square\overset{\overset{2}{\leftarrow}}{\mathrm{F}}\mathrm{F} \geq \overset{\overset{3}{\rightarrow}}{a-\tfrac{7}{2}}$.

Case 1: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{T}\square\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\mathrm{F} \geq \overset{\overset{2}{\rightarrow}}{a-\tfrac{7}{2}}$.

Case 1.1: $\mathrm{T}^{a-2}\mathrm{F}\square\square\mathrm{TTF}\square\mathrm{F} \geq a-\frac{7}{2}$. The left hand side is $\geq (a-2)-1$.

Case 1.2: $\mathrm{T}^{a-2}\square\square\mathrm{TT}\square\mathrm{FF} \geq a-4$. The left hand side is $\geq (a-2)-2$.

Case 2: $\mathrm{T}^{a-2}\mathrm{F}\square\square\mathrm{FTTF}\square\mathrm{F} \geq a-\frac{7}{2}$, true by Lemma B.3.

Case 3: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{T}\square\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\mathrm{F} \geq a-4$.

$\Rightarrow a-4 \geq a-4$, by Lemma B.4. $\square$

Lemma B.6. $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} = a-\frac{7}{2}$, $a \geq 5$.

Proof: Need to show

I) $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} \leq a-\frac{7}{2}$.

II) $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} \geq a-\frac{7}{2}$.

I) To Show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\square\mathrm{T}\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\mathrm{F}\square\square\mathrm{FF} \leq \overset{\overset{3}{\leftarrow}}{a-\tfrac{7}{2}}$.

Case 1: $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{FF} \leq a-\frac{7}{2}$, true by Lemma B.5.

Case 2: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-3}}\square\mathrm{TT}\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\mathrm{F}\square\mathrm{F}\square\mathrm{F} \leq \overset{\overset{3}{\leftarrow}}{a-\tfrac{7}{2}}$.

Case 2.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \leq a-\frac{7}{2}$, true by Lemma B.4.

Case 2.2: $\mathrm{T}^{a-4}\square\mathrm{TTTTFF}\square\square\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is $(a-4)-2$.

Case 2.3: $\mathrm{T}^{a-3}\square\mathrm{TTTFF}\square\square\mathrm{F} \leq a-3$. The left hand side is $(a-3)-2$.

Case 3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\,\square\mathrm{T}\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\mathrm{F}\square\mathrm{F}\square\mathrm{F} \leq a-3$.

Case 3.1: $\mathrm{T}^{a-2}\square\mathrm{TF}\square\mathrm{TF}\square\mathrm{F} \leq a-3$. This is case I)4 of Lemma B.5.

Case 3.2: $\mathrm{T}^{a-3}\square\mathrm{TTTFF}\square\square\mathrm{F} \leq a-3$. The left hand side is $(a-3)-2$.

II) To Show $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\mathrm{F} \geq \overset{\overset{2}{\rightarrow}}{a-\tfrac{7}{2}}$.

Case 1: $\mathrm{T}^{a-2}\square\mathrm{T}\square\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\mathrm{TF}\square\,\overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \overset{\overset{3}{\rightarrow}}{a-\tfrac{7}{2}}$.

Case 1.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TF}\square\mathrm{F} \geq a-\frac{7}{2}$, true by Lemma B.4.

Case 1.2: $\mathrm{T}^{a-2}\square\square\mathrm{TFTFF}\square \geq a-\frac{7}{2}$.
The left hand side is $\geq \mathrm{T}^{a-2}\square\mathrm{F}\square = a-\frac{5}{2}$.

Case 1.3: $\mathrm{T}^{a-2}\square\square\mathrm{TFTF}\square\mathrm{F} \geq a-4$.
The left hand side is $\geq \mathrm{T}^{a-2}\square\mathrm{F}\square - 1 = (a-\frac{5}{2})-1$.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{T}\square\,\overset{\overset{1}{\leftarrow}}{\mathrm{F}}\,\mathrm{T}\square\,\overset{\overset{2}{\leftarrow}}{\mathrm{F}}\,\mathrm{F} \geq a-4$.

Case 2.1: $a-4 \geq a-4$, by Lemma B.5.

Case 2.2: $\mathrm{T}^{a-2}\square\square\mathrm{TFTF}\square\mathrm{F} \geq a-4$. This is case 1.3 above. $\square$

Lemma B.7. $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} = a-\frac{7}{2}$, $a \geq 5$.

Proof: Need to show

I) $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} \leq a-\frac{7}{2}$.

II) $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} \geq a-\frac{7}{2}$.

I) To Show $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\,\square\,\overset{\overset{1}{\rightarrow}}{\mathrm{T}}\,\square\mathrm{F}\square\mathrm{FF} \leq \overset{\overset{3}{\leftarrow}}{a-\tfrac{7}{2}}$.

Case 1: $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\square\mathrm{TFF}\square\mathrm{F} \leq\ \overset{\overset{2}{\leftarrow}}{a} - \frac{7}{2}$.

Case 1.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-2}}\ \square\mathrm{TF}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{F} \leq\ \overset{\overset{3}{\leftarrow}}{a} - \frac{7}{2}$.

Case 1.1.1: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\mathrm{F} \leq\ \overset{\overset{3}{\leftarrow}}{a} - \frac{7}{2}$. (The left hand side is $\{1 \mid \frac{1}{2}\}$)

Case 1.1.1.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\square\square\mathrm{FTF} \leq a-4$.

$\Rightarrow \mathrm{T}^{a-2}\mathrm{F}\square\mathrm{T}\square\mathrm{F}\square\mathrm{TF} \leq a-4$.

$\Rightarrow \mathrm{T}^{a-1}\square\mathrm{F}\square\mathrm{TF} \leq a-4$. The left hand side is 1.

Case 1.1.1.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TFT}\square\square\mathrm{F} \leq a-\frac{7}{2}$.

$\Rightarrow \mathrm{T}^{a-1}\mathrm{FT}\square\square\mathrm{F} \leq a-\frac{7}{2}$. The left hand side is 0.

Case 1.1.1.3: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{FT}\square\square\mathrm{F} \leq a-3$. The left hand side is ≤ 2.

Case 1.1.2: $\mathrm{T}^{a-3}\square\mathrm{TTFTF}\square\square\mathrm{F} \leq a-\frac{7}{2}$.

The left hand side is $\leq (a-3)+\square\mathrm{T}\square\mathrm{F} = (a-3)-\frac{1}{2}$

Case 1.1.3: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{F} \leq a-3$.

Case 1.1.3.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{FT}\square\square\mathrm{F} \leq a-3$. This is case 1.1.1.3 above.

Case 1.1.3.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TTF}\square\square\mathrm{F} \leq a-3$.

$\Rightarrow$ The left hand side is $\leq \mathrm{T}^{a}\mathrm{F}\square\square\mathrm{F} = \{\{1 \mid 1\} \mid 0\}$.

Case 1.2: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\square\mathrm{F} \leq a-3$.

Case 1.2.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{FF}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\square\mathrm{F} \leq a-3$.

Case 1.2.1.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \mathrm{F}\square\mathrm{F}\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F} \leq a-3$.

Case 1.2.1.1.1: $\mathrm{T}^{a-1}\mathrm{FF}\square\square\square\mathrm{TF} \leq a-3$. The left hand side is -2.

Case 1.2.1.1.2: The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} \leq a-3$. The left hand side is $\{\{\{a-3 \mid 2\} \mid 1\} \mid 0\}$.

Case 1.2.1.2: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\square\mathrm{F} \leq a-3$.

Case 1.2.1.2.1: $\mathrm{T}^{a-2}\mathrm{FTF}\square\square\mathrm{T}\square\mathrm{F} \leq a-3$.

$$\Rightarrow \square\mathrm{T}\square\mathrm{T}\square\mathrm{F} = \{1 \mid 1\}.$$

Case 1.2.1.2.2: $\mathrm{T}\square\mathrm{T}\square\square\mathrm{F} \leq a-3$. The left hand side is 2.

Case 1.2.2: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{T}\square\mathrm{F}\square\mathrm{F} \leq a-3$. This is case 1.1.3 above.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} \leq a-\frac{7}{2}$, true by Lemma B.6.

Case 3: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \mathrm{F}\square\square\mathrm{FF} \leq a-3$.

Case 3.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{F}\square\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{FF} \leq a-3$.

Case 3.1.1: $\overset{\overset{2}{\rightarrow}}{\mathrm{T}^{a-1}}\ \square\mathrm{F}\square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F} \leq a-3$.

Case 3.1.1.1: $\mathrm{T}^{a-1}\mathrm{F}\square\square\mathrm{F}\square\mathrm{TF} \leq a-3$.

$$\Rightarrow \mathrm{T}^{a-1}\square\mathrm{F}\square\mathrm{TF} = 1.$$

Case 3.1.1.2: $\mathrm{T}^{a-2}\mathrm{F}\ \overset{\overset{2}{\rightarrow}}{\mathrm{T}}\ \square\square\mathrm{F}\ \overset{\overset{1}{\rightarrow}}{\mathrm{T}}\ \square\mathrm{F} \leq a-3$.

Case 3.1.1.2.1: $\mathrm{T}^{a-2}\mathrm{FT}\square\mathrm{F}\square\square\mathrm{TF} \leq a-3.$

The statement above is true since the left most part of the position gets block eventually.

Case 3.1.1.2.2: $\mathrm{T}^{a-2}\mathrm{F}\square\mathrm{TF}\square\mathrm{T}\square\mathrm{F} \leq a-3.$

The left hand side goes to $\Rightarrow \mathrm{T}^{a-1}\mathrm{F}\square\mathrm{T}\square\mathrm{F} = \{\{a-3 \mid 2\} \mid 1\} \mid 0\}$

Case 3.1.2: $a-3 \leq a-3$, by Lemma B.5.

Case 3.2: $a-3 \leq a-3$, by Lemma B.6.

II) To Show $\mathrm{T}^{a-1}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}} \square \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F} \geq a \overset{\overset{3}{\rightarrow}}{-} \frac{7}{2}$.

Case 1: $\mathrm{T}^{a-2}\square\mathrm{TTF}\square\square\mathrm{FF} \geq a-\frac{7}{2}$, true by Lemma B.6.

Case 2: $\mathrm{T}^{a-2}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq a \overset{\overset{3}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.1: $\mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{FTF}\square\mathrm{F} \geq a-\frac{7}{2}$. This is II) case1 of Lemma B.6.

Case 2.2: $\mathrm{T}^{a-2}\square\mathrm{T}\square\mathrm{T} \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{FF}\square \geq a \overset{\overset{2}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.2.1: $\mathrm{T}^{a-3}\square\mathrm{TTFT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square \geq a \overset{\overset{2}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.2.1.1: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TF}\square \geq a \overset{\overset{2}{\rightarrow}}{-} \frac{7}{2}$.

Case 2.2.1.1.1: $\mathrm{T}^{a-3}\square\mathrm{TTFF}\square\mathrm{TF}\square \geq a-3.$

The left hand side is $(a-3)+0 = a-3.$

Case 2.2.1.1.2: $\mathrm{T}^{a-3}\square\mathrm{T}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TFTF}\square \geq a-4.$

$\Rightarrow \mathrm{T}^{a-3}\square\square\mathrm{FTTFTF}\square = a-4.$

The left hand side $\geq \mathrm{T}^{a-3}\square\square\mathrm{F} + \mathrm{FTF}\square = (a-4)+0 = a-4.$

Case 2.2.1.2: $\mathrm{T}^{a-3}\square\mathrm{TTF}\square\mathrm{TFF}\square \geq a-4$.

The left hand side $\geq \mathrm{T}^{a-3}\square + \square\mathrm{TFF}\square = (a-3)+0 = a-3$.

Case 2.2.2: $\mathrm{T}^{a-2}\square\square\mathrm{TTFFF}\square \geq a-4$. The left hand side is $2(a-2)$.

Case 2.3: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{F}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \quad a-4$.

Case 2.3.1: $\mathrm{T}^{a-3}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \mathrm{TF}\square \overset{\overset{2}{\leftarrow}}{\mathrm{F}} \geq \quad a-4$.

Case 2.3.1.1: $a-4 \geq a-4$, by Case II 1.1 of Lemma B.6.

Case 2.3.1.2: $\mathrm{T}^{a-3}\square\mathrm{T}\square\mathrm{TFTFF}\square \geq a-4$.

The left hand side $\Rightarrow \mathrm{T}^{a-3}\square\mathrm{TF}\square = a-3$.

Case 2.3.2: $\mathrm{T}^{a-3}\square\mathrm{TT}\square\mathrm{TFFF}\square \geq a-4$. The left hand side $\geq a-3$.

Case 3: $\mathrm{T}^{a-2}\square\mathrm{TT}\square \overset{\overset{1}{\leftarrow}}{\mathrm{F}}\ \square \overset{\overset{2}{\leftarrow}}{\mathrm{F}}\ \mathrm{F} \geq \quad a-4$.

Case 3.1: $a-4 \geq a-4$, by Lemma B.6.

Case 3.2: $\mathrm{T}^{a-3}\square\mathrm{TTT}\square\mathrm{FF}\square\mathrm{F} \geq a-4$, true by the Case 2.3 $\square$

Theorem B.8. $\mathrm{T}^{a}\square\square\square\mathrm{FFF} = a - \frac{7}{2}$, $a \geq 5$.

Proof: Need to show

I) $\mathrm{T}^{a}\square\square\square\mathrm{FFF} \leq a - \frac{7}{2}$.

II) $\mathrm{T}^{a}\square\square\square\mathrm{FFF} \geq a - \frac{7}{2}$.

I) To Show $\overset{\overset{1}{\rightarrow}}{\mathrm{T}^{a}}\ \square\square\square\mathrm{FFF} \leq \quad a \overset{\overset{2}{\leftarrow}}{-} \frac{7}{2}$.

Case 1: $\mathrm{T}^{a-1}\square\mathrm{T}\square\mathrm{F}\square\mathrm{FF} \leq a - \frac{7}{2}$, true by Lemma B.7.

Case 2: $T^a\square\square F\square FF \leq a-3$.

$\Rightarrow a-3 \leq a-3$, by Lemma B.7.

II) To Show $T^a\square\square\square \overset{\xleftarrow{1}}{F} FF \geq \overset{\xrightarrow{2}}{a-\frac{7}{2}}$.

Case 1: $T^{a-1}\square T\square F\square FFF \geq a-\frac{7}{2}$, true by Lemma B.7.

Case 2: $T^{a-1}\square T\square\square \overset{\xleftarrow{1}}{F} FF \geq a-4$.

$\Rightarrow a-4 \geq a-4$, by Lemma B.7. $\square$

References

[1] Elwyn Berlekamp, John Conway, and Richard Guy, *Winning Ways for your Mathematical Plays*, Academic Press, New York, 1982.
[2] Jeff Erickson, *New Toads and Frogs Results*, Game of No Chance, 1996.
[3] Jesse Hull, personal website, `http://compgeom.cs.uiuc.edu/ ~jeffe/pubs/toads.html` .
[4] Thotsaporn "Aek" Thanatipanonda, Doron Zeilberger, *A Symbolic Finite-state approach for Automated Proving of Theorems in Combinatorial Game Theory*, J. of Difference Equations and Applications 15(2009), 111-118 .
[5] Thotsaporn "Aek" Thanatipanonda, *Further Hopping with Toads and Frogs*, preprint, `http://www.risc.jku.at/people/tthanati/` .
[6] Thotsaporn "Aek" Thanatipanonda, *Library values for Toads and Frogs*, `http://www.risc.jku.at/people/tthanati/FiniteState.html` .